Practice problems for the Second Midterm, Math 241, Fall 2013

Question 1. Let $u(x,t)$ be a solution of the BVP

$$\left| \begin{aligned} u_{tt} &= 4u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u(0,t) &= 0, \\ u(1,t) &= 1, \\ u(x,0) &= \sin(\pi x), \\ u_t(x,0) &= 0. \end{aligned} \right.$$

Find the value of $u_t\left(\frac{3}{4}, 5\right)$.

(A) $\frac{1}{2}$ (B) 1 (C) 0

(D) $-\frac{2}{\pi}$ (E) $-\frac{2}{5\pi}$ (F) $\frac{3}{4} - \frac{2}{5\pi}$

Question 2. Let $u(x,t)$ be the vertical displacement of a vibrating string of infinite length. The string has constant density $\rho = 1$ and tension with constant magnitude $T = 1$. The initial position $u(x,0) = p(x)$ and velocity $u_t(x,0) = v(x)$ are given by $p(x) = 0$ for all x, and

$$v(x) = \begin{cases} 1, & \text{when } 0 < x < 2, \\ 0, & \text{when } x < 0 \text{ or } x > 2. \end{cases}$$

Calculate the total energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} \left(\rho u_t^2 + T u_x^2\right) dx,$$

of the string.

(A) 3 (B) $2/t$ (C) $16 - t^2$
(D) 1 (E) $-8/3$ (F) $6t$

Question 3. Let $u(x,t)$ be the solution of the BVP:

$$u_{tt} = 16u_{xx} \quad \text{for } 0 < x < 1 \text{ and } t > 0$$

satisfying the boundary conditions

$$u(0,t) = u_x(1,t) = 0$$
$$u(x,0) = \sin\left(\frac{5\pi x}{2}\right) \text{ and } u_t(x,0) = 0$$

What is $u\left(\frac{1}{3}, \frac{1}{4}\right)$?

(A) 20π	(B) $5\pi \sin\left(\frac{5\pi}{6}\right)$	(C) $\sin\left(\frac{5\pi}{6}\right)$
(D) $-\pi$	(E) $1 + 20\pi$	(F) 0

Question 4. Let

$$\sum_{n=-\infty}^{\infty} c_n e^{-in\pi x}$$

be the complex form of the Fourier series of the function $f(x) = 1 - x^2$ on the interval $[-1, 1]$. Find the coefficient c_{-3}.

(A) $\dfrac{2}{9\pi^2}$ (B) $\dfrac{20}{9\pi^2}$ (C) $\dfrac{200}{27\pi^3}$

(D) $\dfrac{1000}{27\pi^3}$ (E) $\dfrac{100}{81\pi^4}$ (F) $\dfrac{2000}{81\pi^4}$

Question 5. Consider the Sturm-Liouville equation

$$\phi" + x\phi + \lambda\phi = 0$$

for a function $\phi(x)$ defined for $0 \leq x \leq \pi$ with boundary conditions

$$\phi'(0) = 0,$$
$$\phi'(\pi) = 0.$$

Let $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the set of of all eigenvalues of the above equation, and let $\phi_n(x)$ be the eigenfunction for the eigenvalue λ_n such that $\phi_n(0) = 1$, $n \geq 1$. Which one of the following statements is true? Justify your reasoning.

(A) $\int_0^\pi \phi_n^2(x)dx = 0$ for $n \geq 1$

(B) $\int_0^\pi x\phi_n(x)\phi_m(x)dx = 0$ for all $n \neq m$

(C) If $n \gg 0$, then $\phi_n(x) < 0$ for all $0 < x < \pi$

(D) If $n \gg 0$, then $\phi_n(x) \leq 0$ for all $0 < x < \pi$

(E) If $n \gg 0$, then $|\phi_n(x)| > 0$ for all $0 < x < \pi$

(F) There exist constants a_0, a_1, a_2, $\ldots$, such that the series

$$\sum_{n=0}^{\infty} a_n\phi_n(2)$$

converges to 10.

Question 6. Express the eigenvalue problem

$$\begin{aligned} e^x \phi'' &= -\lambda \phi \quad \text{on}[1, 10] \\ \phi'(1) &= 0 \\ \phi'(10) &= 0 \end{aligned}$$

in the standard Sturm-Liouville form. Give an approximate formula for the eigenvalues, valid as $\lambda \to \infty$.

Question 7. Solve the following problem posed for $-\infty < x < \infty$ and $t > 0$:

PDE: $u_{tt} = 9u_{xx}$

IC: $u(x, 0) = x^2 - 1, \quad \text{and} \quad u_t(x, 0) = 3\cos(x).$

Question 8. Explicitly show that the eigenvalue problem

$$e^{x^2}\phi'' + x\phi' = -\lambda x^2 \phi \quad \text{on} \quad [1,2] \quad \text{with} \quad \phi(1) = \phi(2) = 0,$$

is a regular Sturm-Liouville problem. Write down the orthogonality condition on the eigenfunctions, and an asymptotic expression for the eigenvalues, valid as $\lambda \to \infty$.