Practice problems for the Second Midterm, Math 241, Fall 2013

Question 1. Let $u(x,t)$ be a solution of the BVP

$$\left|\begin{array}{l} u_{tt} = 4u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u(0,t) = 0, \\ u(1,t) = 1, \\ u(x,0) = \sin(\pi x), \\ u_t(x,0) = 0. \end{array}\right.$$

Find the value of $u_t\left(\frac{3}{4}, 5\right)$.

(A) $\frac{1}{2}$ (B) 1 (C) 0

(D) $-\frac{2}{\pi}$ (E) $-\frac{2}{5\pi}$ (F) $\frac{3}{4} - \frac{2}{5\pi}$

Answer 1.
Since this equation has a non-homogeneous boundary condition, the solution will be equilibrium solution plus solution of a homogeneous problem.
Denote the equilibrium as $v(x)$, it satisfies

$$\left|\begin{array}{l} 0 = 4v''(x), \quad 0 < x < 1 \\ v(0) = 0, \\ v(1) = 1, \end{array}\right.$$

and we find $v(x) = x$.

Now let $w(x,t) = u(x,t) - v(x)$, then w satisfies the homogeneous equation

$$\left|\begin{array}{l} w_{tt} = 4w_{xx}, \quad 0 < x < 1,\ t > 0, \\ w(0,t) = 0, \\ w(1,t) = 0, \\ w(x,0) = \sin(\pi x) - x, \\ w_t(x,0) = 0. \end{array}\right.$$

and we get

$$w(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\big(a_n \cos(2n\pi t) + b_n \sin(2n\pi t)\big)$$

where

$$a_n = 2\int_0^1 \big(\sin(\pi x) - x\big)\sin(n\pi x)dx$$

and

$$b_n = 0$$

Now take time derivative of the solution, we get

$$u_t(x,t) = \sum_{n=1}^{\infty} 2n\pi a_n \sin(n\pi x)\sin(2n\pi t)$$

and when $t = 5$, $\sin(2n\pi t) = 0$, so each of the summand above is 0, thus $u_t(x,5) = 0$, and in particular, we know

$$u_t\Big(\frac{3}{4}, 5\Big) = 0$$

so the correct answer is (C).

Question 2. Let $u(x,t)$ be the vertical displacement of a vibrating string of infinite length. The string has constant density $\rho = 1$ and tension with constant magnitude $T = 1$. The initial position $u(x,0) = p(x)$ and velocity $u_t(x,0) = v(x)$ are given by $p(x) = 0$ for all x, and

$$v(x) = \begin{cases} 1, & \text{when } 0 < x < 2, \\ 0, & \text{when } x < 0 \text{ or } x > 2. \end{cases}$$

Calculate the total energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} \left(\rho u_t^2 + T u_x^2\right) dx,$$

of the string.

(A) 3 (B) $2/t$ (C) $16 - t^2$
(D) 1 (E) $-8/3$ (F) $6t$

Answer 2.
First we show the total energy is conserved, notice the function $u(x,t)$ satisfies the equation

$$\rho u_{tt} = T u_{xx}$$

therefore we have

$$\begin{aligned} \frac{d}{dt}E(t) &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}(\rho u_t^2 + T u_x^2) dx \\ &= \int_{-\infty}^{\infty} (\rho u_t u_{tt} + T u_x u_{xt}) dx \\ &= \int_{-\infty}^{\infty} (T u_t u_{xx} + T u_x u_{xt}) dx \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial x}(T u_t u_x) dx \\ &= 0 \end{aligned}$$

So total energy doesn't change over time, and in particular $E(t) = E(0)$, we can compute $E(0)$ from initial conditions:

$$
\begin{aligned}
E(0) &= \frac{1}{2}\int_{-\infty}^{\infty}\big(\rho u_t^2(x,0) + T u_x^2(x,0)\big)dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} p^2(x) + v^2(x)dx \\
&= \frac{1}{2}\int_{0}^{2} 1dx \\
&= 1
\end{aligned}
$$

So the correct answer is (D).

Question 3. Let $u(x,t)$ be the solution of the BVP:

$$u_{tt} = 16u_{xx} \quad \text{for } 0 < x < 1 \text{ and } t > 0$$

satisfying the boundary conditions

$$u(0,t) = u_x(1,t) = 0$$
$$u(x,0) = \sin\left(\frac{5\pi x}{2}\right) \text{ and } u_t(x,0) = 0$$

What is $u\left(\frac{1}{3}, \frac{1}{4}\right)$?

(A) 20π (B) $5\pi \sin\left(\frac{5\pi}{6}\right)$ (C) $\sin\left(\frac{5\pi}{6}\right)$
(D) $-\pi$ (E) $1 + 20\pi$ (F) 0

Answer 3.
We have the general solution to be

$$u(x,t) = \sum_{n=1}^{\infty} \sin\big((n-1/2)\pi x\big)\Big(a_n \cos(4n-2)\pi t + b_n \sin(4n-2)\pi t\Big)$$

according to initial conditions, $a_3 = 1$ while all other coefficients are 0, so

$$u(x,t) = \sin\left(\frac{5\pi x}{2}\right)\cos 10\pi t$$

and

$$u\Big(\frac{1}{3}, \frac{1}{4}\Big) = \sin\left(\frac{5\pi}{6}\right)\cos\frac{5\pi}{2} = 0$$

the correct answer is (F).

Question 4. Let

$$\sum_{n=-\infty}^{\infty} c_n e^{-in\pi x}$$

be the complex form of the Fourier series of the function $f(x) = 1 - x^2$ on the interval $[-1, 1]$. Find the coefficient c_{-3}.

(A) $\frac{2}{9\pi^2}$ (B) $\frac{20}{9\pi^2}$ (C) $\frac{200}{27\pi^3}$

(D) $\frac{1000}{27\pi^3}$ (E) $\frac{100}{81\pi^4}$ (F) $\frac{2000}{81\pi^4}$

Answer 4.
By formula of complex Fourier coefficients, we have

$$\begin{aligned}
c_n &= \frac{1}{2}\int_{-1}^{1}(1-x^2)e^{n\pi i x}dx \\
&= \frac{1}{2}\int_{-1}^{1}(1-x^2)\frac{1}{n\pi i}d\Big(e^{n\pi i x}\Big) \\
&= \frac{1}{2}(1-x^2)\frac{1}{n\pi i}e^{n\pi i x}\Big|_{-1}^{1} - \frac{1}{2n\pi i}\int_{-1}^{1}e^{n\pi i x}d(1-x^2) \\
&= 0 - 0 - \frac{1}{2n\pi i}\int_{-1}^{1}(-2x)e^{-n\pi x}dx \\
&= \frac{1}{n\pi i}\int_{-1}^{1}\frac{x}{n\pi i}d\Big(e^{n\pi i x}\Big) \\
&= \frac{1}{n\pi i}\frac{x}{n\pi i}e^{n\pi i x}\Big|_{-1}^{1} - \frac{1}{n^2\pi^2 i^2}\int_{-1}^{1}e^{n\pi i x}dx \\
&= \frac{1}{n^2\pi^2 i^2}e^{n\pi i} + \frac{1}{n^2\pi^2 i^2}e^{-n\pi i} - \frac{1}{n^3\pi^3 i^3}\left(e^{n\pi i} - e^{-n\pi i}\right)
\end{aligned}$$

Now plug in $n = -3$, and notice $i^2 = -1$, $e^{3\pi i} = e^{-3\pi i} = -1$, we get

$$c_{-3} = \frac{2}{9\pi^2}$$

So the correct answer is (A).

Question 5. Consider the Sturm-Liouville equation

$$\phi" + x\phi + \lambda\phi = 0$$

for a function $\phi(x)$ defined for $0 \leq x \leq \pi$ with boundary conditions

$$\phi'(0) = 0,$$
$$\phi'(\pi) = 0.$$

Let $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the set of of all eigenvalues of the above equation, and let $\phi_n(x)$ be the eigenfunction for the eigenvalue λ_n such that $\phi_n(0) = 1$, $n \geq 1$. Which one of the following statements is true? Justify your reasoning.

(A) $\displaystyle\int_0^\pi \phi_n^2(x)dx = 0$ for $n \geq 1$

(B) $\displaystyle\int_0^\pi x\phi_n(x)\phi_m(x)dx = 0$ for all $n \neq m$

(C) If $n \gg 0$, then $\phi_n(x) < 0$ for all $0 < x < \pi$

(D) If $n \gg 0$, then $\phi_n(x) \leq 0$ for all $0 < x < \pi$

(E) If $n \gg 0$, then $|\phi_n(x)| > 0$ for all $0 < x < \pi$

(F) There exist constants a_0, a_1, a_2, $\ldots$, such that the series

$$\sum_{n=0}^{\infty} a_n\phi_n(x)$$

converges to 10.

Answer 5.
We see this is a regular Sturm-Liouville problem, with $p(x) = 1$, $q(x) = x$ and $\sigma(x) = 1$. (A) is not true because we are integrating a square of a non-zero continuous function, and we cannot get zero from it.

(B) is not true: according to Sturm-Liouville theorem, we have the orthogonality conditions to be

$$\int_0^\pi \phi_n(x)\phi_m(x)dx = 0, \quad n \neq m$$

if in addition, we have

$$\int_0^\pi x\phi_n(x)\phi_m(x)dx = 0, \quad n \neq m$$

then $\phi_n(x)$ and $x\phi_n(x)$ must be linearly dependent, because ϕ_n form a complete orthonomal basis, but $c\phi_n(x) = x\phi_n(x)$ has no solution for any constant c if ϕ_n is a non-zero continuous function, so (B) cannot be true.
(C),(E) are not true because $\phi_n(x)$ has exactly $n-1$ zeros in the interval $[0,\pi]$, so it cannot be always negative or always positive, and therefore $|\phi_n|$ cannot be always positive as well.
(D) is not true because when n large enough, ϕ_n is asymptotic to some cosine function, which is oscilating, and cannot always be non-positive.
(F) is true because ϕ_n forms a complete basis, for any continuous function f of period π, we are able to expand it under such a basis, and the expansion converges to the function itself because the f is continuous. If in particular, we take the function f to be constantly 10, and have the expansion

$$10 \sim \sum_{n=1}^\infty a_n\phi_n(x)$$

then we can compute the coefficients by orthogonality:

$$a_n = \frac{\int_0^\pi 10\phi_n(x)dx}{\int_0^\pi \phi_n^2(x)dx}$$

then the convergence property tells us

$$10 = \sum_{n=1}^\infty a_n\phi_n(x)$$

So only statement (F) is true.

Question 6. Express the eigenvalue problem

$$\begin{aligned} e^x\phi'' &= -\lambda\phi \quad \text{on}[1, 10] \\ \phi'(1) &= 0 \\ \phi'(10) &= 0 \end{aligned}$$

in the standard Sturm-Liouville form. Give an approximate formula for the eigenvalues, valid as $\lambda \to \infty$.

Answer 6.

Divide by e^x, and we can identify the equation with standard Sturm-Liouville form, with $p(x) = 1$, $q(x) = 0$, and $\sigma(x) = e^{-x}$. Then the large eigenvalues λ_n's are asymptoticly

$$\lambda_n \sim \left(\frac{n\pi}{\int_1^{10} \left(\frac{\sigma}{p}\right)^{\frac{1}{2}} dx}\right)^2 = \left(\frac{n\pi}{\int_1^{10} e^{-\frac{x}{2}} dx}\right)^2 = \left(\frac{n\pi}{2e^{-\frac{1}{2}} - 2e^{-5}}\right)^2$$

Question 7. Solve the following problem posed for $-\infty < x < \infty$ and $t > 0$:

$$\text{PDE:} \quad u_{tt} = 9u_{xx}$$

$$\text{IC:} \quad u(x,0) = x^2 - 1, \quad \text{and} \quad u_t(x,0) = 3\cos(x).$$

Answer 7.
By d'Alembert's formula

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big((x-3t)^2 - 1 + (x+3t)^2 - 1\Big) + \frac{1}{6}\int_{x-3t}^{x+3t} 3\cos s ds \\ &= \frac{1}{2}\Big((x-3t)^2 + (x+3t)^2 - 2\Big) + \frac{1}{6}\Big(-3\sin(x+3t) + 3\sin(x-3t)\Big) \\ &= x^2 + 9t^2 - 1 + \frac{1}{2}\Big(3\sin(x-3t) - 3\sin(x+3t)\Big) \end{aligned}$$

Question 8. Explicitly show that the eigenvalue problem

$$e^{x^2}\phi'' + x\phi' = -\lambda x^2\phi \quad \text{on} \quad [1,2] \quad \text{with} \quad \phi(1) = \phi(2) = 0,$$

is a regular Sturm-Liouville problem. Write down the orthogonality condition on the eigenfunctions, and an asymptotic expression for the eigenvalues, valid as $\lambda \to \infty$.

Answer 8.
In order to identify it with standard Sturm-Liouville problem, we have to multiply the equation by a integrating factor, let that be $f(x)$, so the equation becomes

$$f(x)e^{x^2}\phi'' + f(x)x\phi' + \lambda f(x)x^2\phi = 0$$

comparing with standard form

$$p(x)\phi'' + p'(x)\phi' + q(x)\phi + \lambda\sigma(x)\phi = 0$$

we need the coefficient of ϕ' term to be the derivative of the coefficient of ϕ'' term, i.e.

$$\left(e^{x^2}f(x)\right)' = xf(x)$$

and we get the equation

$$e^{x^2}f' + 2xe^{x^2}f = xf$$

separate x and f, we get

$$\frac{f'}{f} = \frac{x - 2xe^{x^2}}{e^{x^2}} = xe^{-x^2} - 2x$$

the left hand side is $(\ln f)'$, so we get

$$\ln f = \int_0^x \left(se^{-s^2} - 2s\right)ds = -\frac{1}{2}e^{-x^2} - x^2$$

and

$$f(x) = e^{-\frac{1}{2}e^{-x^2} - x^2}$$

therefore we can identify this equation with standard Sturm-Liouville equation by letting

$$p(x) = e^{x^2}f(x) = e^{-\frac{1}{2}e^{-x^2}}, \quad q(x) = 0, \quad \sigma(x) = f(x)x^2 = x^2e^{-\frac{1}{2}e^{-x^2} - x^2}$$

Since exponential functions are continuous, and non-negative, also the boundary conditions satisfiy the requirement, so this is a regular Sturm-Liouville problem. The orthogonality condition is

$$\int_1^2 \phi_m(x)\phi_n(x)x^2e^{-\frac{1}{2}e^{-x^2} - x^2}dx = 0, \quad m \neq n$$

and large eigenvalues are asymptotic to

$$\lambda_n \sim \left(\frac{n\pi}{\int_1^2 \left(\frac{\sigma(x)}{p(x)}\right)^{\frac{1}{2}} dx}\right)^2 = \left(\frac{n\pi}{\int_1^2 xe^{-\frac{x^2}{2}} dx}\right)^2 = \left(\frac{n\pi}{e^{-\frac{1}{2}} - e^{-2}}\right)^2$$
