Practice problems for the Final Exam, Math 241, Fall 2013

This collection of problems is intended to give you practice problems that are comparable in format and difficulty to those which will appear in the coming final exam. The questions in the actual exam will be <u>**DIFFERENT**</u>.

Question 1. Let $u(x,t)$ be the concentration of a chemical per unit volume, and satisfy the following initial and boundary value problem:

$$\left|\begin{array}{l} \dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2} + \dfrac{3}{25}x^2 \quad \text{for } 0 < x < 5, t > 0 \\ \dfrac{\partial u}{\partial x}(0,t) = 0 \\ \dfrac{\partial u}{\partial x}(5,t) = 1 \\ u(x,0) = \dfrac{1}{10}x^2 \end{array}\right.$$

Denote the total amount/mass of the chemical by

$$M(t) := \int_0^5 u(x,t)\,dx.$$

Answer the following questions:

(i) What is the physical meaning of the boundary condition $\frac{\partial u}{\partial x}(5,t) = 1$?

(ii) Compute $\frac{dM}{dt}$.

(iii) Compute M.

Question 2. Solve the wave equation of $u(r, \theta, t)$ on a membrane shaped as a 45^o circular sector of radius 1:

$$
\left|
\begin{aligned}
u_{tt} &= \Delta u, \quad 0 < r < 1, \ \ 0 < \theta < \frac{\pi}{4} \\
u(r, 0, t) &= 0 \\
u_\theta\left(r, \frac{\pi}{4}, t\right) &= 0 \\
u(1, \theta, t) &= 0 \\
u(r, \theta, 0) &= f(r, \theta) \\
u_t(r, \theta, 0) &= 0.
\end{aligned}
\right.
$$

Question 3. Decide whether the following statements regarding the Fourier series are correct or not.

(i) The Fourier cosine series of an odd function is always odd. Y / N

(ii) The Fourier series of $f(x) := x$ is bounded. Y / N

(iii) The coefficients of the Fourier sine series of a bounded function are always bounded. Y / N

Question 4. The function $u(r,\theta)$ describes the steady state temperature distribution in a thin plate R shaped as an anulus with outer radius 2 and inner radius 1. Suppose that heat flux across the boundary of R is given by $u_r(1,\theta) = 2$ for the inner circle, and $u_r(2,\theta) = c\sin^2(3\theta)$ for the outer circle.

(a) What must the value of the constant c be? That is: what must the value of c be so that the boundary value problem

$$\left|\begin{array}{l} \nabla^2 u = 0 \\ u_r(1,\theta) = 2 \\ u_r(2,\theta) = c\sin^2(3\theta) \end{array}\right.$$

will have a solution.

(b) Find the solution $u(r,\theta)$ corresponding to the value of c that you found in part (a).

Question 5. Consider the following eigenvalue problem

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + 2\dfrac{d\phi}{dx} + \lambda e^{2x}\phi = 0 \quad \text{for } 0 < x < 1 \\ \phi(0) = 0 \\ \dfrac{d\phi}{dx}(1) + 2\phi(1) = 0, \end{cases}$$

and answer the following questions.

(i) Rewrite the ordinary differential equation into the Sturm-Liouville form.

(ii) Are all eigenvalues $\lambda \geq 0$?

(iii) Estimate the *large* eigenvalues.

Question 6. Solve the Laplace equation on the interior of a sphere of radius π centered at the origin, subject to the boundary condition $u(\pi, \theta, \phi) = \cos(3\phi)$.

Question 7. Solve the following initial and boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 1 - \dfrac{1}{2}x + tx + t\sin \pi x & \text{for } 0 < x < 2,\ t > 0 \\ u(0,t) = t \\ u(2,t) = t^2 \\ u(x,0) = 3\sin 4\pi x. \end{cases}$$

Question 8. Use the Fourier transform in x to solve the initial value problem

$$\left| \begin{aligned} u_t &= 2u_x - u, \\ u(x,0) &= 7x \end{aligned} \right.$$

Question 9. Solve

$$\begin{cases} \nabla^2 u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = r \qquad \text{for } r < 2,\ -\pi \le \theta \le \pi \\ \qquad\qquad\qquad\qquad\qquad u(2,\theta) = 16\cos 3\theta. \end{cases}$$