Practice problems for the Final Exam, Math 241, Fall 2013

This collection of problems is intended to give you practice problems that are comparable in format and difficulty to those which will appear in the coming final exam. The questions in the actual exam will be **DIFFERENT**.

Question 1. Let $u(x,t)$ be the concentration of a chemical per unit volume, and satisfy the following initial and boundary value problem:

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} + \frac{3}{25}x^2 \qquad \text{for } 0 < x < 5, t > 0 \tag{1}$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \tag{2}$$

$$\frac{\partial u}{\partial x}(5,t) = 1 \tag{3}$$

$$u(x,0) = \frac{1}{10}x^2. \tag{4}$$

Denote the total amount/mass of the chemical by

$$M(t) := \int_0^5 u(x,t)\,dx.$$

Answer the following questions:

(i) What is the physical meaning of the boundary condition $\frac{\partial u}{\partial x}(5,t) = 1$?

(ii) Compute $\frac{dM}{dt}$.

(iii) Compute M.

Answer 1. (i) According to the Fick's law of diffusion, the flux at the right endpoint (i.e., $x = 5$) is

$$\phi(5, t) = -k\frac{\partial u}{\partial x}(5, t) = -k < 0.$$

Therefore, the boundary condition $\frac{\partial u}{\partial x}(5, t) = 1$ means that the atoms of the chemical migrate into the one-dimensional region ($0 < x < 5$) from its right endpoint.

(ii) Using (1) - (3), we have

$$\begin{aligned}\frac{dM}{dt} &= \frac{d}{dt}\int_0^5 u(x, t)\ dx \\ &= \int_0^5 \frac{\partial u}{\partial t}(x, t)\ dx \\ &= \int_0^5 4\frac{\partial^2 u}{\partial x^2}(x, t) + \frac{3}{25}x^2\ dx \\ &= \left[4\frac{\partial u}{\partial x}\right]_{x=0}^5 + \left[\frac{1}{25}x^3\right]_{x=0}^5 \\ &= 4 + 5 \\ &= 9.\end{aligned}$$

(iii) Since $\frac{dM}{dt} = 9$, we have

$$M(t) = M(0) + 9t.$$

In order to compute $M(0)$, we use the initial condition (4) as follows:

$$\begin{aligned}M(0) &= \int_0^5 u(x, 0)\ dx \\ &= \int_0^5 \frac{1}{10}x^2\ dx \\ &= \left[\frac{1}{30}x^3\right]_{x=0}^5 \\ &= \frac{25}{6}.\end{aligned}$$

Therefore,

$$M(t) = \frac{25}{6} + 9t.$$

Question 2. Solve the wave equation of $u(r,\theta,t)$ on a membrane shaped as a 45^o circular sector of radius 1:

$$\left|\begin{aligned} & u_{tt} = \Delta u, \quad 0 < r < 1, \ \ 0 < \theta < \frac{\pi}{4} \\ & u(r,0,t) = 0 \\ & u_\theta\left(r, \frac{\pi}{4}, t\right) = 0 \\ & u(1,\theta,t) = 0 \\ & u(r,\theta,0) = F(r,\theta) \\ & u_t(r,\theta,0) = 0. \end{aligned}\right.$$

Answer 2. Let $u(r,\theta,t) = f(r)g(\theta)h(t)$, and get the ODE's as

$$\left|\begin{aligned} & h''(t) = -\lambda h(t) \\ & g''(\theta) = -\mu g(\theta) \\ & r\big(rf'(r)\big)' + (\lambda r^2 - \mu)f = 0 \end{aligned}\right.$$

First solve the equation of g, it comes with 2 boundary conditions: $g(0) = 0$ and $g'(\frac{\pi}{4}) = 0$, therefore we have

$$\mu_m = (4m-2)^2, \quad m = 1, 2, \cdots$$

and

$$g_m(\theta) = \sin(4m-2)\theta.$$

Next plug in $\mu_m = (4m-2)^2$ to the equation of f, notice that equation comes with boundary conditions: $f(1) = 0$ and $|f(0)| < +\infty$, so we get

$$f_{m,n}(r) = J_{4m-2}(\sqrt{\lambda_{m,n}}r), \quad n = 1, 2, \cdots$$

and

$$\lambda_{m,n} = z_{4m-2,n},$$

where $z_{4m-2,n}$ is the n-th zero of $J_{4m-2}(z)$. Next plug in $\lambda_{m,n}$ to the equation of h, and we get

$$h_{m,n}(t) = A_{m,n}\cos(\sqrt{\lambda_{m,n}}\ t) + B_{m,n}\sin(\sqrt{\lambda_{m,n}}\ t).$$

So the general solution is

$$u(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big(A_{m,n}\cos(\sqrt{\lambda_{m,n}}\ t) + B_{m,n}\sin(\sqrt{\lambda_{m,n}}\ t)\Big)J_{4m-2}(\sqrt{\lambda_{m,n}}r)\sin(4m-2)\theta$$

According to initial conditions, we have

$$A_{m,n} = \frac{\int_0^{\frac{\pi}{4}} \int_0^1 F(r,\theta) J_{4m-2}(\sqrt{\lambda_{m,n}} r) \sin(4m-2)\theta \ r dr d\theta}{\int_0^{\frac{\pi}{4}} \int_0^1 J_{4m-2}^2(\sqrt{\lambda_{m,n}} r) \sin^2(4m-2)\theta \ r dr d\theta}$$

and

$$B_{m,n} = 0$$

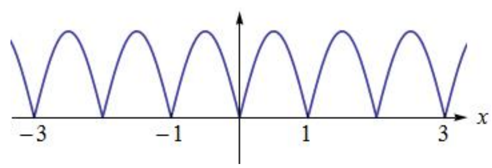
Question 3. Decide whether the following statements regarding the Fourier series are correct or not.

(i) The Fourier cosine series of an odd function is always odd. Y / N

(ii) The Fourier series of $f(x) := x$ is bounded. Y / N

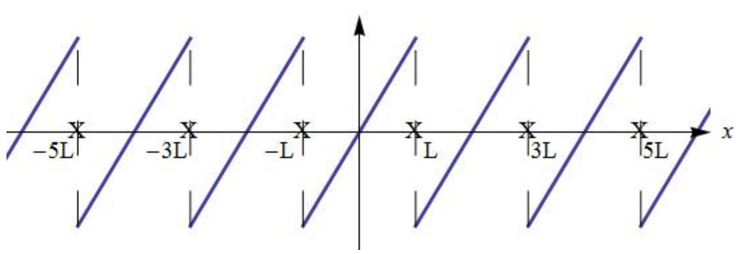
(iii) The coefficients of the Fourier sine series of a bounded function are always bounded. Y / N

Answer 3. (i) No! A counter example is $f(x) := \sin \pi x$, which is an odd function. According to the procedure stated on page 104 of the textbook, one can sketch its Fourier cosine series on $[0, 1]$ as follows:



Indeed, the Fourier cosine series of f is even instead of odd.

(ii) Yes! Let us sketch the Fourier series of $f(x) := x$ on $[-L, L]$ as follows:



According to the above graph, the Fourier series of $f(x) := x$ is bounded.

(iii) Yes! Assume that f is bounded, i.e., there exists a constant $M > 0$ such that

$$|f(x)| \leq M.$$

Suppose that $f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$. Then

$$\begin{aligned} |b_n| &:= \left| \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \; dx \right| \\ &\leq \frac{2}{L} \int_0^L |f(x)| \; dx \\ &\leq \frac{2}{L} \int_0^L M \; dx \\ &= 2M. \end{aligned}$$

Question 4. The function $u(r,\theta)$ describes the steady state temperature distribution in a thin plate R shaped as an anulus with outer radius 2 and inner radius 1. Suppose that heat flux across the boundary of R is given by $u_r(1,\theta) = 2$ for the inner circle, and $u_r(2,\theta) = c\sin^2(3\theta)$ for the outer circle.

(a) What must the value of the constant c be? That is: what must the value of c be so that the boundary value problem

$$\left| \begin{aligned} \nabla^2 u &= 0 \\ u_r(1,\theta) &= 2 \\ u_r(2,\theta) &= c\sin^2(3\theta) \end{aligned} \right.$$

will have a solution.

(b) Find the general solution $u(r,\theta)$, you don't need to compute the coefficients.

Answer 4. (a) In order for this system to reach its steady state, we need the net heat flux to be zero, i.e.,

$$\int_0^{2\pi} u_r(1,\theta) - 2u_r(2,\theta)\ d\theta = 0$$

so we have

$$2c\int_0^{2\pi} \sin^2(3\theta)\ d\theta = \int_0^{2\pi} 2\ d\theta = 4\pi$$

and

$$\int_0^{2\pi} \sin(3\theta) = \int_0^{2\pi} \frac{1-\cos(6\theta)}{2}\ d\theta = \pi$$

therefore $c = 2$.

(b) Let $u(r,\theta) = f(r)g(\theta)$, then we get the ODE's:

$$\left| \begin{aligned} g''(\theta) &= -\lambda g(\theta) \\ r\big(rf'(r)\big)' &= \lambda f(r) \end{aligned} \right.$$

For the equation of g, it comes with periodic boundary conditions:

$$g(-\pi) = g(\pi), \quad g'(-\pi) = g'(\pi)$$

so we know

$$\lambda_n = n^2, \quad n = 0, 1, 2, \cdots$$

and

$$g_0 = A_0$$

$$g_n(\theta) = A_n \cos n\theta + B_n \sin n\theta, \quad n \geq 1$$

Next plug in $\lambda_n = n^2$ to the equation of f, we get

$$f_0(r) = C_0 + D_0 \ln r$$

$$f_n(r) = C_n r^n + D_n r^{-n}, \quad n \geq 1$$

So the general solution is

$$u(r, \theta) = (C_0 + D_0 \ln r) + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n})(A_n \cos n\theta + B_n \sin n\theta)$$

Question 5. Consider the following eigenvalue problem

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + 2\dfrac{d\phi}{dx} + \lambda e^{2x}\phi = 0 \quad \text{for } 0 < x < 1 \\ \phi(0) = 0 \\ \dfrac{d\phi}{dx}(1) + 2\phi(1) = 0, \end{cases}$$

and answer the following questions.

(i) Rewrite the ordinary differential equation into the Sturm-Liouville form.

(ii) Are all eigenvalues $\lambda \geq 0$?

(iii) Estimate the *large* eigenvalues.

Answer 5. (i) Multiplying the differential equation by e^{2x}, we have

$$e^{2x}\frac{d^2\phi}{dx^2} + 2e^{2x}\frac{d\phi}{dx} + \lambda e^{4x}\phi = 0.$$

Using the fact that

$$\frac{d}{dx}\left(e^{2x}\frac{d\phi}{dx}\right) = e^{2x}\frac{d^2\phi}{dx^2} + 2e^{2x}\frac{d\phi}{dx},$$

we obtain

$$\frac{d}{dx}\left(e^{2x}\frac{d\phi}{dx}\right) + \lambda e^{4x}\phi = 0,$$

which is in the Sturm-Liouville form

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0$$

with $p(x) = e^{2x}$, $q(x) \equiv 0$, and $\sigma(x) = e^{4x}$.

(ii) Since the eigenvalue problem is a *regular* Sturm-Liouville eigenvalue problem, we can apply the Rayleigh quotient to study its eigenvalues:

$$\begin{aligned} \lambda &= \frac{-p\phi\frac{d\phi}{dx}\Big|_{x=0}^{1} + \int_0^1 p\left|\frac{d\phi}{dx}\right|^2 - q\phi^2\, dx}{\int_0^1 \phi^2\sigma\, dx} \\ &= \frac{2e^2|\phi(1)|^2 + \int_0^1 e^{2x}\left|\frac{d\phi}{dx}\right|^2\, dx}{\int_0^1 \phi^2 e^{4x}\, dx} \\ &\geq 0. \end{aligned}$$

Remark 1 **($\lambda > 0$)** *Indeed, one can also prove that all eigenvalue λ must be positive. The reasoning is as follows.*

Seeking for a contradiction, suppose that $\lambda = 0$ is an eigenvalue. Due to the Rayleigh quotient

$$0 = \lambda = \frac{2e^2|\phi(1)|^2 + \int_0^1 e^{2x} \left|\frac{d\phi}{dx}\right|^2 \, dx}{\int_0^1 \phi^2 e^{4x} \, dx},$$

we know that the corresponding eigenfunction ϕ must satisfy

$$\int_0^1 e^{2x} \left|\frac{d\phi}{dx}\right|^2 \, dx = 0 \tag{5}$$

$$|\phi(1)| = 0 \tag{6}$$

since both terms in the numerator of the Rayleigh quotient are non-negative. Since the integrand $e^{2x} \left|\frac{d\phi}{dx}\right|^2$ is non-negative, (5) implies

$$\frac{d\phi}{dx} \equiv 0,$$

which implies

$$\phi \equiv \text{ constant.}$$

By (6),

$$\phi \equiv 0,$$

which contradicts with the fact that ϕ is an eigenfunction. Therefore, $\lambda = 0$ is not an eigenvalue. Combining this fact with what we have already proved in part (ii), we know that all eigenvalues $\lambda > 0$.

(iii) According to the WKB theory, when the eigenvalue λ is large, the eigenfunction ϕ can be approximated by

$$\begin{aligned}\phi(x) &\approx (\sigma p)^{-\frac{1}{4}} \sin\left(\sqrt{\lambda} \int_\alpha^x \left(\frac{\sigma}{p}\right)^{\frac{1}{2}} \, dx_0\right) + \cdots \\ &\approx e^{-\frac{3x}{2}} \sin\left(\sqrt{\lambda} \int_\alpha^x e^{x_0} \, dx_0\right) + \cdots\end{aligned}$$

where α is a parameter that will be determined by the boundary condition, and the symbol $\cdots$ represents the lower terms that the reader should ignore at this moment.

Using the boundary condition $\phi(0) = 0$, we have

$$0 \approx \sin\left(\sqrt{\lambda} \int_\alpha^0 e^{x_0} \, dx_0\right) + \cdots .$$

If we want to choose α so that the above identity holds and α is independent of λ, then the only possible choice will be

$$\alpha = 0.$$

That is,

$$\phi(x) \approx e^{-\frac{3x}{2}} \sin\left(\sqrt{\lambda} \int_0^x e^{x_0}\ dx_0\right) + \cdots .$$

Using the boundary condition $\frac{d\phi}{dx}(1) + 2\phi(1) = 0$, we have

$$\begin{aligned} 0 &\approx \sqrt{\lambda} e^{-\frac{1}{2}x} \cos\left(\sqrt{\lambda} \int_0^x e^{x_0}\ dx_0\right)\Big|_{x=1} + 2e^{-\frac{3x}{2}} \sin\left(\sqrt{\lambda} \int_0^x e^{x_0}\ dx_0\right)\Big|_{x=1} + \cdots \\ &\approx \sqrt{\lambda} e^{-\frac{1}{2}} \cos\left(\sqrt{\lambda} \int_0^1 e^{x_0}\ dx_0\right) + 2e^{-\frac{3}{2}} \sin\left(\sqrt{\lambda} \int_0^1 e^{x_0}\ dx_0\right) + \cdots , \end{aligned}$$

and hence,

$$\tan\left(\sqrt{\lambda} \int_0^1 e^{x_0}\ dx_0\right) \approx -\frac{\sqrt{\lambda} e}{2} + \cdots .$$

Since λ is very large, the right hand side can be approximated by $-\infty$. Therefore,

$$\sqrt{\lambda} \int_0^1 e^{x_0}\ dx_0 \approx n\pi + \frac{\pi}{2}$$

where n is any sufficiently large integer. Hence, the large eigenvalue λ can be approximated by

$$\lambda \approx \left(\frac{n\pi + \frac{\pi}{2}}{\int_0^1 e^{x_0}\ dx_0}\right)^2 = \left(\frac{n\pi + \frac{\pi}{2}}{e - 1}\right)^2$$

for any sufficiently large integer n.

Question 6. Solve the Laplace equation on the interior of a sphere of radius π centered at the origin, subject to the boundary condition $u(\pi, \theta, \phi) = \cos(3\phi)$.

Answer 6. Let $u(\rho, \theta, \phi) = f(\rho)g(\theta)h(\phi)$, we get the ODE's

$$\left| \begin{array}{l} g''(\theta) = -\mu g(\theta) \\ \big(\sin\phi\ h'(\phi)\big)' + \Big(\lambda \sin\phi - \dfrac{\mu}{\sin\phi}\Big) h(\phi) = 0 \\ \big(\rho^2 f'(\rho)\big)' - \lambda f(\rho) = 0 \end{array} \right.$$

For the equation of g, it comes with the periodic boundary conditions $g(-\pi) = g(\pi)$, $g'(-\pi) = g'(\pi)$, so

$$\mu_m = m^2, \quad m = 0, 1, 2, \cdots$$

and

$$g_m(\theta) = A_m \cos m\theta + B_m \sin m\theta.$$

Plug in $\mu_m = m^2$ to the equation of h, together with the boundedness requirements $|h(0)| < +\infty$ and $|h(\pi)| < +\infty$, we get

$$\lambda_n = n(n+1), \quad n \geq m,$$

and

$$h(\phi) = P_n^m(\cos\phi).$$

Next plug in $\lambda_n = n(n+1)$ to the equation of f, and according to boundedness at the origin: $|f(0)| < +\infty$, we get

$$f(\rho) = \rho^n$$

therefore the general solution is

$$u(\rho, \theta, \phi) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \Big(A_{m,n} \rho^n P_n^m(\cos\phi) \cos m\theta + B_{m,n} \rho^n P_n^m(\cos\phi) \sin m\theta \Big)$$

Notice the boundary condition $u(\pi, \theta, \phi) = \cos(3\phi)$ doesn't depend on θ, therefore we throw away the terms containning θ, i.e., the terms with $m \geq 1$, so

$$u(\rho, \theta, \phi) = \sum_{n=0}^{\infty} A_n \rho^n P_n^0(\cos\phi)$$

and

$$\cos(3\phi) = u(\pi, \theta, \phi) = \sum_{n=3}^{\infty} A_n \pi^n P_n^0(\cos\phi)$$

hence by orthogonality, we get

$$A_n \pi^n = \frac{\int_0^\pi \cos(3\phi) P_n^0(\cos\phi) \ \sin\phi d\phi}{\int_0^\pi \left(P_n^0(\cos\phi)\right)^2 \sin\phi d\phi}$$

Question 7. Solve the following initial and boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 1 - \dfrac{1}{2}x + tx + t\sin \pi x & \text{for } 0 < x < 2,\ t > 0 \\ u(0,t) = t \\ u(2,t) = t^2 \\ u(x,0) = 3\sin 4\pi x. \end{cases}$$

Answer 7. Let $r(x,t) := t - \frac{1}{2}tx + \frac{1}{2}t^2x$, and $v(x,t) := u(x,t) - r(x,t)$. Then v satisfies

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} + t\sin \pi x \\ v(0,t) = 0 \\ v(2,t) = 0 \\ v(x,0) = 3\sin 4\pi x. \end{cases} \tag{7}$$

To solve (7), we apply the eigenfunction expansion

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{2}. \tag{8}$$

Substituting (8) into the equation $(7)_1$, we have

$$\sum_{n=1}^{\infty} a_n' \sin \frac{n\pi x}{2} = -\sum_{n=1}^{\infty} \frac{n^2\pi^2}{4} a_n \sin \frac{n\pi x}{2} + t\sin \pi x$$
$$\sum_{n=1}^{\infty} \left\{ a_n' + \frac{n^2\pi^2}{4} a_n \right\} \sin \frac{n\pi x}{2} = t\sin \pi x.$$

Comparing the coefficients, we have

$$a_n' + \frac{n^2\pi^2}{4} a_n = \begin{cases} t & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Solving these ordinary differential equations, we obtain

$$a_n(t) = \begin{cases} \left(a_2(0) + \frac{1}{\pi^4}\right) e^{-\pi^2 t} + \frac{\pi^2 t - 1}{\pi^4} & \text{if } n = 2 \\ a_n(0) e^{-\frac{n^2\pi^2}{4} t} & \text{otherwise.} \end{cases} \tag{9}$$

Now, using the initial condition $v(x,0) = 3\sin 4\pi x$, we have

$$3\sin 4\pi x = \sum_{n=1}^{\infty} a_n(0) \sin \frac{n\pi x}{2}.$$

Comparing the coefficients, we have

$$a_n(0) = \begin{cases} 3 & \text{if } n = 8 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Comparing (9) and (10), we finally obtain

$$a_n(t) = \begin{cases} \frac{1}{\pi^4} e^{-\pi^2 t} + \frac{\pi^2 t - 1}{\pi^4} & \text{if } n = 2 \\ 3e^{-16\pi^2 t} & \text{if } n = 8 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by (8),

$$v(x,t) = \left(\frac{1}{\pi^4} e^{-\pi^2 t} + \frac{\pi^2 t - 1}{\pi^4} \right) \sin \pi x + 3e^{-16\pi^2 t} \sin 4\pi x,$$

and hence,

$$\begin{aligned} u(x,t) &= v(x,t) + r(x,t) \\ &= \left(\frac{1}{\pi^4} e^{-\pi^2 t} + \frac{\pi^2 t - 1}{\pi^4} \right) \sin \pi x + 3e^{-16\pi^2 t} \sin 4\pi x + t - \frac{1}{2} tx + \frac{1}{2} t^2 x. \end{aligned}$$

Question 8. Use the Fourier transform in x to solve the initial value problem

$$\left|\begin{aligned} u_t &= 2u_x - u, \\ u(x,0) &= 7x \end{aligned}\right.$$

Answer 8. Apply Fourier transform on x, and denote

$$U(w,t) = \mathcal{F}[u]$$

then $U(w,t)$ satisfies

$$\left|\begin{aligned} &\frac{\partial}{\partial t}U(w,t) = -2iwU(w,t) - U(w,t) \\ &U(w,0) = \mathcal{F}[7x] \end{aligned}\right.$$

Therefore we have

$$U(w,t) = U(w,0)e^{(-2iw-1)t} = \mathcal{F}[7x]e^{(-2iw-1)t}$$

Apply inverse Fourier transform, we have

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[U(w,t)] \\ &= \mathcal{F}^{-1}[\mathcal{F}[7x]e^{(-2iw-1)t}] \\ &= \frac{1}{2\pi}\mathcal{F}^{-1}[\mathcal{F}[7x]] * \mathcal{F}^{-1}[e^{(-2iw-1)t}] \\ &= \frac{1}{2\pi}(7x) * \mathcal{F}^{-1}[e^{(-2iw-1)t}] \\ &= \frac{1}{2\pi}e^{-t}(7x) * \mathcal{F}^{-1}[e^{-2iwt}] \\ &= \frac{1}{2\pi}e^{-t}(7x) * (2\pi\delta_{-2t}(x)) \\ &= e^{-t}\int_{-\infty}^{\infty} 7(x-y)\delta_{-2t}(y)\ dy \\ &= e^{-t}7(x+2t) \\ &= 7(x+2t)e^{-t}. \end{aligned}$$

Question 9. Solve

$$\begin{cases} \nabla^2 u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = r \quad \text{for } r < 2,\ -\pi \le \theta \le \pi \\ u(2,\theta) = 16\cos 3\theta. \end{cases}$$

Answer 9. First of all, let us decompose $u := u_1 + u_2$ where u_1 and u_2 satisfy

$$\begin{cases} \nabla^2 u_1 = r \\ u_1(2,\theta) = 0 \\ |u_1(0,\theta)| < \infty \end{cases} \qquad \text{and} \qquad \begin{cases} \nabla^2 u_2 = 0 \\ u_2(2,\theta) = 16\cos 3\theta \\ |u_2(0,\theta)| < \infty. \end{cases}$$

Following the computations in subsection 2.5.2, one may solve $\nabla^2 u_2 = 0$ in the disk by the general solution formula

$$u_2(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta.$$

We will skip the derivation of this solution formula here, but you ***MUST*** *provide the skipped details in the exam to show your understanding.*

Using the boundary condition of u_2, we have

$$16\cos 3\theta = \sum_{n=0}^{\infty} 2^n A_n \cos n\theta + \sum_{n=1}^{\infty} 2^n B_n \sin n\theta.$$

Comparing coefficients, we have

$$A_n = \begin{cases} 2 & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad B_n \equiv 0.$$

Therefore,

$$u_2(r,\theta) = 2r^3 \cos 3\theta.$$

To solve u_1, we observe that due to the symmetry of the equation and boundary condition, we can assume that u_1 is independent of θ, i.e., $u_1 = u_1(r)$. Therefore, we can solve u_1 as follows:

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_1}{\partial r}\right) &= r \\ \frac{\partial}{\partial r}\left(r\frac{\partial u_1}{\partial r}\right) &= r^2 \\ r\frac{\partial u_1}{\partial r} &= \frac{1}{3}r^3 + c_1 \\ \frac{\partial u_1}{\partial r} &= \frac{1}{3}r^2 + \frac{c_1}{r} \\ u_1 &= \frac{1}{9}r^3 + c_1 \ln r + c_2. \end{aligned}$$

Using the boundary conditions of u_1, we have

$$c_1 = 0 \text{ and } c_2 = -\frac{8}{9}.$$

Therefore,

$$u_1(r, \theta) = \frac{r^3 - 8}{9}.$$

As a result, the final answer is

$$\begin{aligned} u(r, \theta) &= u_1(r, \theta) + u_2(r, \theta) \\ &= \frac{r^3 - 8}{9} + 2r^3 \cos 3\theta. \end{aligned}$$
