Practice problems for the First Midterm, Math 241, Fall 2013

Question 1. Let $u(x, y)$ be a function of two variables satisfying the Laplace equation

$$u_{xx} + u_{yy} = 0.$$

Suppose that u is a radial function, that is: in polar coordinates, the function u depends on the coordinate r but is indepenendant of the coordinate θ. Suppose also that $u(1, 0) = 0$ and $u(0, e) = 1/2$. What is the value of $u(e^2, 0)$?

(A) $e/2$ (B) $e + 1$ (C) 1
(D) e (E) e^2 (F) $e^2 + 1$

Answer 1. In polar coordinates (r, θ) the two dimensional Laplace equation reads

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

If u is a function that only depends on r, then $\partial^2 u/\partial\theta = 0$. Hence $u(r)$ is a solution of the ODE BVP:

$$\left|\begin{array}{l} \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{du}{dr}\right) = 0, \\ u(1) = 0, \\ u(e) = 1/2. \end{array}\right.$$

Multiplying both sides of the equation by r and integrating we get

$$\int \frac{d}{dr}\left(r\frac{du}{dr}\right) dr = a$$

for some constant a. Thus $du/dr = a/r$ and integrating once again we get

$$u(r) = a\ln(r) + b,$$

for some constants a and b. Now from the boundary conditions we get

$$0 = a\ln(1) + b = b, \qquad \text{and} \qquad \frac{1}{2} = a\ln(e) = a.$$

Hence

$$u(r) = \frac{1}{2}\ln(r),$$

and since the point $(e^2, 0)$ has polar coordinates $r = e^2$, $\theta = 0$ we get

$$u(e^2, 0) = \frac{1}{2}\ln(e^2) = 1.$$

The correct answer is (C). □

Question 2. Heat is flowing through a thin wire of length 2 meters, so that one end of the wire is at $x = 0$ and the other end is at $x = 2$. Let $u(x,t)$ be the temeperature at point x and time t, and suppose $u(x,t)$ satisfies the inhomogeneous heat equation

$$u_t = \frac{1}{4}u_{xx} + 4 - x^2, \qquad \text{for } 0 < x < 2, \text{ and } t > 0.$$

If the boundary conditions are

$$u(0,t) = 0 \quad \text{and} \quad u_x(2,t) = 0,$$

then what is

$$\lim_{t\to+\infty} u(2,t),$$

that is what is the equilibrium temperature at the insulated end of the wire?

(A) 0	(B) 1/2	(C) 16
(D) 4	(E) 8/3	(F) 6

Answer 2. Since we are looking for a value of the equilibrium temperature we can solve the problem by finding the equilibrium solution and evaluating it. The equilibrium temperature $U(x) := \lim_{t\to+\infty} u(x,t)$ will satisfy

$$\frac{1}{4}U\text{"}(x) + 4 - x^2, \qquad \text{for} 0 \le x \le 2$$

and the boundary conditions

$$U(0) = 0 \qquad \text{and} \qquad U'(2) = 0.$$

Integrating twice we get

$$U(x) = -8x^2 + \frac{1}{3}x^4 + c_1 x + c_2,$$

where c_1 and c_2 are arbitrary constants. The boundary condition $U(0) = 0$ implies $c_2 = 0$, while $U'(2) = 0$ gives

$$0 = U'(2) = -32 + \frac{32}{3} + c_1,$$

or $c_1 = 64/3$.

Therefore

$$U(x) = -8x^2 + \frac{1}{3}x^4 + \frac{64}{3}x$$

and

$$\lim_{t+\infty} u(2,t) = U(2) = -32 + \frac{16}{3} + \frac{128}{3} = 16.$$

The correct answer is (C).

Different solution: We can find all solutions to the given BVP. The heat equation is inhomogeneous and so, before we can use separation of variables, we have to reduce to the homogeneous case.

For this it suffices to find some particular solution $s(x,t)$ of the inhomogeneous equation. Indeed if s is such solution, and if $u(x,t)$ is the function we are interested in, then for the difference $v(x,t) = u(x,t) - s(x,t)$ we get

$$v_t = u_t - s_t = \left(\frac{1}{4}u_{xx} + 4 - x^2\right) - \left(\frac{1}{4}s_{xx} + 4 - x^2\right) = \frac{1}{4}(u_{xx} - s_{xx}) = \frac{1}{4}v_{xx}.$$

In other words v will satisfy the homogeneous equation

$$v_t = \frac{1}{4}v_{xx},$$

and the boundary conditions

$$v(0,t) = -s(0,t) \quad \text{and} \quad v_x(2,t) = -s_x(2,t).$$

The right hand side of the inhomogeneous equation depends on x only, which suggests that we should look for $s(x,t)$ which depends only on x. If $s = s(x)$, then the inhomogeneous equation reads

$$0 = \frac{1}{4}s''(x) + 4 - x^2.$$

Integrating once we get

$$s'(x) = \int(-16 + 4x^2)dx = -16x + \frac{4}{3}x^3 + c,$$

and integrating a second time we get

$$s(x) = \int \left(-16x + \frac{4}{3}x^3 + c\right) dx = -8x^2 + \frac{1}{3}x^4 + cx + d.$$

For the separation of variables we need homogeneous boundary conditions so we choose c and d so that $s(0) = s'(2) = 0$. This gives $d = 0$, $c = 64/3$ and

$$s(x) = -8x^2 + \frac{1}{3}x^4 + \frac{64}{3}x.$$

With such a choice of particular solution the function v will satisfy

$$\left| \begin{aligned} v_t &= \frac{1}{4}v_{xx}, \\ v(0,t) &= 0, \\ v_x(2,t) &= 0. \end{aligned} \right.$$

We first look for separated solutions

$$v(t,x) = G(t)\varphi(x).$$

Substituting in the equation we get

$$\frac{dG}{dt} \cdot \varphi = \frac{1}{4}G\frac{d^2\varphi}{dx^2},$$

and so we will have

$$\frac{4}{G}\frac{dG}{dt} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = \mu$$

for some constant μ.

Furthermore, the boundary conditions imply $\varphi(0) = \varphi'(2) = 0$ and so φ must solve

$$\left| \begin{aligned} \varphi'' &= \lambda\varphi, \\ \varphi(0) &= 0, \\ \varphi'(2) &= 0. \end{aligned} \right.$$

Considering the various possibilities for λ we se that we must have

$$\lambda = -\left(\frac{(2n+1)\pi}{4}\right)^2, \qquad n = 0, 1, 2, \ldots$$

with the corresponding φ given by

$$\varphi(x) = c \cdot \sin\left(\frac{(2n+1)\pi x}{4}\right).$$

By the principle of superposition we conclude that

$$v(x,t) = \sum_{n=0}^{\infty} a_n \sin\left(\frac{(2n+1)\pi x}{4}\right) \exp\left(-\left(\frac{(2n+1)\pi}{4}\right)^2 \frac{t}{4}\right),$$

and so

$$\begin{aligned} v(2,t) &= \sum_{n=0}^{\infty} a_n \sin\left(\frac{(2n+1)\pi}{2}\right) \exp\left(-\left(\frac{(2n+1)\pi}{4}\right)^2 \frac{t}{4}\right) \\ &= \sum_{n=0}^{\infty} a_n (-1)^n \exp\left(-\left(\frac{(2n+1)\pi}{4}\right)^2 \frac{t}{4}\right). \end{aligned}$$

But for every $n \geq 0$ we have

$$\lim_{t\to+\infty} \exp\left(-\left(\frac{(2n+1)\pi}{4}\right)^2 \frac{t}{4}\right) = 0,$$

so

$$\lim_{t\to+\infty} v(2,t) = 0,$$

and hence

$$\lim_{t\to+\infty} u(2,t) = \lim_{t\to+\infty} (v(2,t) + s(2)) = 0 + 16 = 16.$$

The correct answer is (C). □

Question 3. Let $u(x,t)$ be the solution of the equation:

$$u_t = u_{xx} \quad \text{for } 0 < x < 4 \text{ and } t > 0$$

satisfying the boundary conditions

$$\begin{aligned} u(0,t) = 0 &\text{ and } u(4,t) = 0 \\ u(x,0) &= 3\sin 8\pi x \end{aligned}$$

What is $u\left(\frac{3}{16}, 3\right)$?

(A) $3e^{-\pi^2}$ (B) $3e^{-64\pi^2}$ (C) $3e^{-192\pi^2}$
(D) $-3e^{-\pi^2}$ (E) $-3e^{-64\pi^2}$ (F) $-3e^{-192\pi^2}$

Answer 3. This is a heat equation with homogeneous Dirichlet boundary conditions at the ends of the interval. From the separation of variables analysis we carried out in class, we know that any function of the form

$$u(x,t) = \sum_{n=1}^{+\infty} a_n \sin\left(\frac{n\pi x}{4}\right) \exp\left(-\left(\frac{n\pi}{4}\right)^2 t\right)$$

will solve the equation and will satisfy the homogeneous boundary conditions. Thus we only need to choose the coefficients a_n so that

$$3\sin 8\pi x = u(x,0) = \sum_{n=1}^{+\infty} a_n \sin\left(\frac{n\pi x}{4}\right).$$

Since the Fourier coefficients are uniquely determined by the orthogonality of the sine functions and $\sin 8\pi x$ appears in the right hand side as the term corresponding to $n = 32$, we must have that $a_n = 0$ for $n \neq 32$, and $a_{32} = 3$.

In other words

$$u(x,t) = 3\sin(8\pi x)e^{\left(-64\pi^2 t\right)}.$$

Evaluating at $x = 3/16$, $t = 3$ we get

$$u\left(\frac{3}{16}, 3\right) = 3\sin\left(\frac{3\pi}{2}\right)e^{-192\pi^2} = -3e^{-192\pi^2}.$$

The correct answer is (F). □

Question 4. If, for $0 \leq x \leq 5$, we have

$$\sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi x}{5}\right) = 5x - x^2,$$

what is the value of b_3?

(A) $\frac{2}{9\pi^2}$ (B) $\frac{20}{9\pi^2}$ (C) $\frac{200}{27\pi^3}$

(D) $\frac{1000}{27\pi^3}$ (E) $\frac{100}{81\pi^4}$ (F) $\frac{2000}{81\pi^4}$

Answer 4. From the formula for the coefficients of the Fourier sine series we compute

$$
\begin{aligned}
b_3 &= \frac{2}{5}\int_0^5 (5x - x^2)\sin\left(\frac{3\pi x}{5}\right) dx \\
&= -\frac{2}{3\pi}\int_0^5 (5x - x^2) d\cos\left(\frac{3\pi x}{5}\right) \\
&= -\frac{2}{3\pi}(5x - x^2)\cos\left(\frac{3\pi x}{5}\right)\Bigg|_0^5 + \frac{2}{3\pi}\int_0^5 \cos\left(\frac{3\pi x}{5}\right) d(5x - x^2) \\
&= \frac{2}{3\pi}\int_0^5 \cos\left(\frac{3\pi x}{5}\right)(5 - 2x)dx \\
&= \frac{10}{9\pi^2}\int_0^5 (5 - 2x) d\sin\left(\frac{3\pi x}{5}\right) \\
&= \frac{10}{9\pi^2}(5 - 2x)\sin\left(\frac{3\pi x}{5}\right)\Bigg|_0^5 - \frac{10}{9\pi^2}\int_0^5 \sin\left(\frac{3\pi x}{5}\right) d(5 - 2x) \\
&= \frac{20}{9\pi^2}\int_0^5 \sin\left(\frac{3\pi x}{5}\right) dx \\
&= -\frac{100}{27\pi^3}\cos\left(\frac{3\pi x}{5}\right)\Bigg|_0^5 \\
&= -\frac{100}{27\pi^3}((-1) - 1) \\
&= \frac{200}{27\pi^3}.
\end{aligned}
$$

The correct answer is (C). □

Question 5. Suppose the heat flux at every point of the outer circle of an annulus with inner radius $R_1 = 2$ and outer radius $R_2 = 5$ points directly out of the annulus and has magnitude 4. Also, suppose that at every point of the inner circle of the annulus the flux points directly into the annulus and has the same magnitude all around the circle. What must the magnitude of this latter flux be so that the temperature of the annulus will be at equilibrium? (In other words, so that there is a solution of the Laplace equation with these flux values.)

(A) -4 (B) 0 (C) 4
(D) -10 (E) 10 (F) 8

Answer 5. In polar coordinates a point (r,θ) belongs to the annulus if $2 \le r \le 5$ and $-\pi \le \theta \le \pi$. Since at the points of the outer circle $r = 5$ the heat flux points outside and in the radial direction it follows that the heat flux is computed by the partial derivative $\partial u/\partial r$. Similarly, at the points of the inner circle $r = 2$, the heat flux points inside the annulus in the radial direction, and so it is again computed by the partial derivative $\partial u/\partial r$. Write c for the constant value of the heat flux at the points of the inner circle. Then the boundary conditions given by the flux conditions are

$$u_r(5,\theta) = 4, \qquad u_r(2,\theta) = c.$$

Taking into account the fact that the temperature and the heat flux have to be continuous inside the annulus we get the following boundary value problem on $u(r,\theta)$:

$$\left|\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \\ u(r,-\pi) = u(r,\pi) \\ u_\theta(r,-\pi) = u_\theta(r,\pi) \\ u_r(5,\theta) = 4 \\ u_r(2,\theta) = c \end{aligned}\right. \tag{1}$$

We look for a product solution

$$u(r,\theta) = G(r)\varphi(\theta).$$

The variables in the PDE separate after dividing with $(1/r^2)G\varphi$, and we get

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\varphi}\frac{d^2\varphi}{d\theta^2} = \lambda.$$

Requiring that this product solution of the Laplace equation satisfies the periodic boundary conditions in (1) gives the ODE boundary value problem

$$\left|\begin{aligned} \frac{d^2\varphi}{d\theta^2} = -\lambda\varphi, \\ \varphi(-\pi) = \varphi(\pi), \\ \frac{d\varphi}{d\theta}(-\pi) = \frac{d\varphi}{d\theta}(\pi). \end{aligned}\right. \tag{2}$$

This problem has a solution only when $\lambda = n^2$, $n = 0, 1, 2, \ldots$ Furthermore

- if φ is a solution of (2) corresponding to $\lambda = n^2 \neq 0$, then φ is a linear combination of $\cos(n\theta)$ and $\sin(n\theta)$;
- if φ is a solution of (2) corresponding to $\lambda = 0$, then φ is constant.

If we also require that the product solution $u = G(r)\varphi(\theta)$ satisfies the remaining conditions $u_r(5,\theta) = 4$ and $u_r(2,\theta) = c$, then we have to use a $\varphi(\theta)$ which is constant, i.e. we have to be in the case $\lambda = 0$.

For $\lambda = 0$, the equation on G becomes the separable ODE

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0.$$

Dividing both sides by r/G and integrating we get

$$G(r) = c_1 + c_2 \ln(r).$$

Thus for $\lambda = 0$ we get a product solution $u(r,\theta) = c_1 + c_2\ln(r)$ of the Laplace equation which by construction satisfies the homogeneous boundary conditions $u(r,-\pi) = u(r,\pi)$, $u_\theta(r,-\pi) = u_\theta(r,\pi)$. To satisfy the boundary condition $u_r(5,\theta) = 4$ we must choose c_1 and c_2 so that

$$4 = u_r(5,\theta) = \frac{c_2}{5}.$$

This gives $c_2 = 20$ and so $u(r,\theta) = c_1 + 20\ln(r)$.

In particular

$$u_r(r,\theta) = \frac{20}{r}$$

and so

$$c = u_r(2,\theta) = 10.$$

The correct answer is (E).

Different solution: We can use conservation of energy to determine the value of c. Since the equilibrium temeperature u satisfies the Laplace equation $\nabla^2 u = 0$ on the annulus $R = \{(r,\theta) \mid 2 \leq r \leq 5,\ -\pi \leq \theta \leq \pi\}$ we have that

$$\iint_R \left(\nabla^2 u\right) dA = 0.$$

Computing this integral in polar coordinates we get

$$
\begin{aligned}
0 &= \iint_R \left(\nabla^2 u\right) dA \\
&= \int_{\theta=-\pi}^{\pi} \int_{r=2}^{5} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) r dr d\theta \\
&= \int_{\theta=-\pi}^{\pi} \left[\int_{r=2}^{5} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \right) r dr \right] d\theta + \int_{r=2}^{5} \left[\int_{\theta=-\pi}^{\pi} \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} d\theta \right] r dr \\
&= \int_{\theta=-\pi}^{\pi} \left[\int_{r=2}^{5} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) dr \right] d\theta + \int_{r=2}^{5} \left[\int_{\theta=-\pi}^{\pi} \frac{\partial^2 u}{\partial \theta^2} d\theta \right] \frac{1}{r} dr \\
&= \int_{\theta=-\pi}^{\pi} \left[\left(r \frac{\partial u}{\partial r} \right) \Big|_{r=2}^{5} \right] d\theta + \int_{r=2}^{5} \left[\frac{\partial u}{\partial \theta} \Big|_{\theta=-\pi}^{\pi} \right] \frac{1}{r} dr \\
&= \int_{\theta=-\pi}^{\pi} \left[5u_r(5,\theta) - 2u_r(2,\theta) \right] d\theta + \int_{r=2}^{5} \left[u_\theta(r,\pi) - u_\theta(r,-\pi) \right] \frac{1}{r} dr \\
&= \int_{\theta=-\pi}^{\pi} (20 - 2c)\, d\theta + \int_{r=2}^{5} 0 \cdot \frac{1}{r} dr \\
&= 2\pi(20 - 2c).
\end{aligned}
$$

Therefore we must have $c = 10$ and the correct answer is (E). $\square$

Question 6.

(a) Compute the Fourier cosine series for the function $f(x) = x^2$ on the interval $[0, \pi]$.

(b) Fully simplify your answer - the formula for the coefficients should not involve sines or cosines.

(c) Does the Fourier cosine series converge to the function f at the point $x = 0$? Justify your answer.

Answer 6. **(a)** The Fourier cosine series of $f(x) = x^2$ is

$$\sum_{n=0}^{\infty} a_n \cos(nx), \tag{3}$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{\pi} x^2 dx, \\ a_n &= \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx) dx, \qquad \text{for } n > 0. \end{aligned}$$

(b) To simplify the formulas for the coefficients we compute the integrals. For $n = 0$ we have

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{\pi^2}{3},$$

and for $n > 0$ we have

$$\begin{aligned} &= \frac{2}{n\pi}\int_0^{\pi} x^2 d\sin(nx) \\ &= \frac{2}{n\pi} x^2 \sin(nx)\Big|_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} x\sin(nx) dx \\ &= \frac{4}{n^2\pi}\int_0^{\pi} x d\cos(nx) \\ &= \frac{4}{n^2\pi} x d\cos(nx)\Big|_0^{\pi} \\ &= \frac{4}{n^2\pi}\pi d\cos(n\pi) \\ &= \frac{(-1)^n 4}{n^2}. \end{aligned}$$

Thus the Fourier cosine series of x^2 is

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{n^2} \cos(nx).$$

(c) Yes, the Fourier series of $f(x) = x^2$ converges to $f(0) = 0$ at $x = 0$. This follows from the convergence theorem and the fact that the even extension of x^2 on $[-\pi, \pi]$ is continuous at $x = 0$. The last statement is clear since the even extension of f is given by the formula $f(x) = x^2$ for all $-\pi \leq x \leq \pi$. $\square$

Question 7. Consider the following BVP posed for $0 < x < L$ and $t > 0$:

$$\text{PDE:} \quad u_t = u_{xx} + 2u_x + u$$

$$\text{BC:} \quad u(0,t) = 0, \quad \text{and} \quad u(L,t) + u_x(L,t) = 0$$

Apply the method of separation of variables to determine what ordinary differential equations are implied for functions of x and t and what boundary conditions (if any) are necessary for each of those ODEs. You do **not** need to solve these ODEs.

Answer 7. We look for a special product solution of the form $u(x,t) = G(t)\varphi(x)$. Substituting the product in the PDE we get

$$\frac{dG}{dt}\varphi = G\frac{d^2\varphi}{dx^2} + 2G\frac{d\varphi}{dx} + G\varphi.$$

Dividing both sides by $G\varphi$ separates the variables:

$$\frac{1}{G}\frac{dG}{dt} = \frac{1}{\varphi}\left(\frac{d^2\varphi}{dx^2} + 2\frac{d\varphi}{dx} + \varphi\right).$$

Since the left hand side in this equation is a function of t only, the right hand side is a function of t only, and x and t are independent variables, it follows that both sides must be constant. In other words, there exists some constant μ so that

$$\frac{1}{G}\frac{dG}{dt} = \frac{1}{\varphi}\left(\frac{d^2\varphi}{dx^2} + 2\frac{d\varphi}{dx} + \varphi\right) = \mu.$$

These gives two ODE - one for $G(t)$ and one for $\varphi(x)$. Explicitly the ODE are:

$$\frac{dG}{dt} = \mu G$$

$$\frac{d^2\varphi}{dx^2} + 2\frac{d\varphi}{dx} + (1-\mu)\varphi = 0.$$

The boundary conditions on u give boundary conditions for $\varphi(x)$. Indeed, to have $u(0,t)=0$ for all $t>0$ we must to have $G(t)\varphi(0)=0$ for all $t>0$. Thus we will either have $\varphi(0)=0$ or $G(t)=0$ for all $t>0$. But if $G(t)$ is identically zero, then $u(x,t)$ will be identically zero, i.e.will be a trivial solution. Thus, to have a non-trivial solution we must have $\varphi(0)=0$. Similarly, to have $u(L,t)+u_x(L,t)=0$ we must have $G(t)(\varphi(L)+\varphi'(L))=0$, and so for a non-trivial solution we must have $\varphi(L)+\varphi'(L)=0$. In summary, for $G(t)$ we get the ODE

$$G'(t)=\mu G(t)$$

with no initial or boundary conditions, while for $\varphi(x)$ we get the BVP

$$\left|\begin{aligned} \varphi\text{"}+2\varphi+(1-\mu)\varphi=0\\ \varphi(0)=0\\ \varphi(L)+\varphi'(L)=0.\end{aligned}\right.$$

□

Question 8. Show that for $0<x<1$ and $t>0$, the function

$$u(x,t)=e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{1}{2}}$$

is a solution of the heat equation

$$u_t=u_{xx}.$$

Give one example of a homogeneous boundary condition that this solution satisfies on the specified domain.

Answer 8. We compute

$$\begin{aligned} u_t &= \frac{\partial}{\partial t}\left(e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{1}{2}}\right)\\ &= \frac{x^2}{4(t+1)^2}e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{1}{2}}+e^{-\frac{x^2}{4t+4}}\cdot\left(-\frac{1}{2}\right)\cdot(t+1)^{-\frac{3}{2}} \qquad (4)\\ &= \frac{x^2}{4}e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{5}{2}}-\frac{1}{2}e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{3}{2}}.\end{aligned}$$

and

$$\begin{aligned} u_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{1}{2}}\right)\right) \\ &= \frac{\partial}{\partial x}\left(-\frac{2x}{4t+4}e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{1}{2}}\right) \\ &= -\frac{2}{4t+4}e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{1}{2}} + \frac{4x^2}{(4t+4)^2}e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{1}{2}} \\ &= -\frac{1}{2}e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{3}{2}} + \frac{x^2}{4}e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{5}{2}}. \end{aligned} \tag{5}$$

Together (4) and (5) show that $u_t = u_{xx}$, i.e. that u satisfies the heat equation.

Also, in the first step in (5) we got that

$$u_x(x,t) = -\frac{2x}{4t+4}e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{1}{2}}.$$

Thus the heat flux at $x = 0$ vanishes, i.e. $u(x,t)$ satisfies the homogeneous boundary condition $u_x(0,t) = 0$ for all $t \geq 0$. $\square$
