Math 314, HOMEWORK 1

Topics this week:

- Vector spaces - definition and first properties (Section 1.1)
- Fields, examples, Euclid's algorithm
- Examples of vector spaces (Section 1.1)
- Constructions, spans, and llinear conditions (Section 1.2)

Problems - to be turned in on January 31, 2020

1. Which of the following are fields. If so, explain why all the axioms hold. If not, point out an axiom that fails:

- the positive real numbers $(\mathbb{R}_{>0}, +, .)$ with the natural addition and multiplication operations;
- the non-negative real numbers $(\mathbb{R}_{\geq 0}, +, .)$ with the natural addition and multiplication operations;
- all polynomials $\mathbb{Q}[x]$ in one variable with rational coefficients;
- the power set $(\mathcal{P}(S), \cup, \cap)$ of a given set S, with the operations of union and intersection;
- the power set $(\mathcal{P}(S), \Delta, \cap)$ of a given set S, with the operations of symmetric difference and intersection;
- the integers modulo 17 $(\mathbb{Z}/17, +, .)$ with the natural addition and multiplication operations;
- the integers modulo 21 $(\mathbb{Z}/21, +, .)$ with the natural addition and multiplication operations;
- the set $(\mathbb{Q}[i], +, .)$ of all complex numbers of the form $a + bi$, $a, b \in \mathbb{Q}$ with the natural addition and multiplication operations.

2. Determine whether each of the following is a vector space over the indicated f ield. If so, explain why all the axioms hold. If not, point out which axiom fails:

- The set of real numbers $\mathbb{R}$ over the field of rational numbers $\mathbb{Q}$ with addition in $\mathbb{R}$ defined to be the usual addition, and the $\mathbb{Q}$-scaling in $\mathbb{R}$ defined to be the usual multiplication of a real number by a rational number.
- The set $\mathrm{Mat}_{2\times 3}(\mathbb{R})$ of all 2×3 real matrices over the field of real numbers $\mathbb{R}$. Addition and $\mathbb{R}$-scaling of matrices are defined entry by entry.
- The set $\mathbb{R}[x]$ of all polynomials with real coefficients over the field of rational numbers $\mathbb{Q}$. Addition and $\mathbb{Q}$-scaling of polynomials are defined coefficient by coefficient.
- The set of integrs $\mathbb{Z}/8$ over the Boolean field $\mathbb{F}_2$ with addition and scaling defined as the usual addition and multiplication of numbers modulo 8.

3. Let S be a set and let $\mathrm{Fun}(S, \mathbb{R})$ be the $\mathbb{R}$-vector space of functions from S to $\mathbb{R}$, with addition and scaling defined value by value. Which of the following subsets in $\mathrm{Fun}(S, \mathbb{R})$ are $\mathbb{R}$-vector spaces with respect to the same addition and scaling operations. In each case explain while all vector space axioms hold or point out an axiom that fails.

- the subset of all functions equal to zero at a given point $s \in S$;
- the subset of all functions $f : S \to \mathbb{R}$ such that $f(s) = 0$ for all points s in a given subset $T \subset S$;
- the subset of all functions $f : S \to \mathbb{R}$ such that $f(s) = 1$ for all points s in a given subset $T \subset S$;
- the subset of all funtions $f : S \to \mathbb{R}$ such that there exists a point $s \in S$ with $f(s) = 0$;
- If $S = \mathbb{R}$, the subset of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{s\to+\infty} f(s) = 0$.