Math 371, practice problems for exam 3

1. Let $A \neq 0$ be a unital commutative ring.

(a) View A as an A-module and let $x, y \in A$ be any two elements. Show that x and y must be linearly dependent over A.

(b) Suppose that every ideal in A is free as an A-module. Show that A is a PID.

2. Let $R = \mathbb{Z}[1/2] \subset \mathbb{Q}$ and consider the matrix $A \in \mathrm{Mat}_{2\times 3}(R)$ given by

$$A = \begin{pmatrix} 14 & 2 & 3/2 \\ -6 & 0 & -10 \end{pmatrix}.$$

(a) Prove that R is a PID.

(b) Use invertible row and column operations over R to diagonalize the matrix A.

3. Prove the following facts using the Eistenstein criterion.

(a) $\sqrt{2}$ is not a rational number.

(b) $x^6 + 4x^3 + 1$ is irreducible in $\mathbb{Q}[x]$.

(c) $\mathbb{C}[x, y, z, w]/(xw - yz)$ is an integral domain.

4. Let R be a domain. View R as a module over itself. Show that the module R is isomorphic to every non-zero submodule $M \subset R$ if and only if R is a PID.

5. Let R be a PID and let A, B, and C be finitely generated R-modules. Show that $A \oplus B \cong A \oplus C$ implies $B \cong C$.

6. Let M be the finitely generated abelian group obtained as the quotient of $\mathbb{Z}^{\oplus 3}$ by the subgoup N spanned by the elements x_1, x_2, x_3, where

$$\left| \begin{array}{l} x_1 = 7e_1 + 2e_2 + 3e_3, \\ x_2 = 21e_1 + 8e_2 + 9e_3, \\ x_3 = 5e_1 - 4e_2 + 3e_3. \end{array} \right.$$

Find the decomposition of M into a direct sum of a free abelian group and primary cylic groups.

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7. Let $A \subset B$ be finitely generated abelian groups. Show that $\operatorname{rank}(B/A) = \operatorname{rank}(B) - \operatorname{rank}(A)$.

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8. Let R be a ring. An R-module M is called *irreducible* if $M \neq 0$ and the only submodules of M are 0 and M.

(a) Show that M is irreducible if and only if $M \neq 0$ and M is cyclic with any non-zero element as generator.

(b) Find all irreducible $\mathbb{Z}$-modules.

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9. Let $A \neq 0$ be a unital commutative ring. View A as module over itself via multiplication. Prove that $\operatorname{End}_A(A)$ and A are isomorphic as rings.

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10. Let $V = \mathbb{C}^2$ be the two dimensional coordinate space over $\mathbb{C}$, and let $f : V \to V$ be the linear operator given by $f(x, y) = (0, y)$. Consider V as a module over the polynomial ring $\mathbb{C}[t]$ via the action $p(t) \cdot v = p(f)(v)$ for any $p(t) \in \mathbb{C}[t]$ and any $v \in V$. Find all $\mathbb{C}[t]$-submodules in V.
