Math 503 Spring 2015
Practice Problems for the Final

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(1) Prove that if R is a division ring, then $\mathrm{Mat}_{n\times n}(R)$ has no nontrivial two-sided ideals.

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(2) Let R be a ring. Prove that the center of the ring $\mathrm{Mat}_{n\times n}(R)$ consists of all matrices of the form $a \cdot I_n$, where I_n is the identity matrix, and a belongs to the center of R.

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(3) (i) Suppose that G is a finite group and k is a field. Prove that $k[G]^{op} \cong k[G]$.

(ii) Let $\mathbb{H}$ be the division ring of all quaternions, i.e. the subring of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ consisting of matrices of the form $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$. Prove that $\mathbb{H}^{op} \cong \mathbb{H}$.

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(4) Let $A = \mathbb{Z} \oplus \mathbb{Z}i \subset \mathbb{C}$ be the subring of Gaussian integers.

(a) Prove that $A/(2)$ is not a field.

(b) Prove that $A/(3)$ is a field.

(5) Find all $a \in \mathbb{F}_7$ for which $\mathbb{F}_7[x]/(x^2 + a)$ is a field.

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(6) Let R be a commutative ring with the property that every left R-module is free. Show that R is a field.

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(7) Let A be a commutative ring and let M be an A-module.

(a) Show that M and $\mathrm{Hom}_A(A, M)$ are isomorpic A-modules.

(b) Show that the natural forgetful map $\mathrm{Hom}_A(A, M) \to \mathrm{Hom}_{\mathbb{Z}}(A, M)$ is an inclusion of abelian groups. Give an explicit example of A and M where this map is **not** an isomorphism.

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(8) Let F be a field, let $F[t]$ be the polynomial ring in one variable over F, and let $a \neq b \in F$.

(i) Show that the quotient rings $F[t]/(t-a)$ and $F[t]/(t-b)$ are both isomorphic to F.

(ii) Show that the quotient $F[t]$-modules $F[t]/(t-a)$ and $F[t]/(t-b)$ are **not** isomorphic.

(iii) Prove that the $F[t]$-module $(F[t]/(t-a)) \oplus (F[t]/(t-b))$ is cyclic.

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(9) Let A be an integral domain and let M be a free A-module. Prove that if $x \in M$ and $a \in A$ are such that $ax = 0$, then either $a = 0$ or $x = 0$. Give examples to sho that this is no longer true if you drop either the condition that A is an integral domain or the condition that M is free.

(10) Let $A \subset B$ be finitely generated abelian groups. Show that $\operatorname{rank}(B/A) = \operatorname{rank}(B) - \operatorname{rank}(A)$.

(11) Let R be a PID and let A, B, and C be finitely generated R-modules. Show that $A \oplus B \cong A \oplus C$ implies $B \cong C$.

(12) Let M be the finitely generated abelian group obtained as the quotient of $\mathbb{Z}^{\oplus 3}$ by the subgoup N spanned by the elements x_1, x_2, x_3, where

$$\left| \begin{aligned} x_1 &= 7e_1 + 2e_2 + 3e_3, \\ x_2 &= 21e_1 + 8e_2 + 9e_3, \\ x_3 &= 5e_1 - 4e_2 + 3e_3. \end{aligned} \right.$$

Find the decomposition of M into a direct sum of a free abelian group and primary cylic groups.