Math 503 Fall 2015
Practice Problems for the Midterm

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(1) True or false. Give a reason or a counter-example

(a) The map $\sigma : GL_n(\mathbb{R}) \to \mathrm{Aut}_{\mathrm{Set}}\left(\mathrm{Mat}_{n\times n}(\mathbb{R})\right)$ given by $\sigma_P(A) = PAP^t$ defines an action of $GL_n(\mathbb{R})$ on the set of $n \times n$ matrices.

(b) Let G be a group acting on a set X. Let $H \subset G$ be the subset $H = \{g \in G \mid g \cdot x = x \}$. Then H is a normal subgroup of G.

(c) If $n \geq 3$, then S_n has trivial center.

(d) If G is a group of odd order and $x \neq e \in G$, then x can not be conjugate to x^{-1}.

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(2) Determine the automorphism group of a cyclic group of order 10.

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(3) Find all finite groups that have exactly two conjugacy classes.

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(4) Let B be the group of invertible upper-triangular $n \times n$ matrices. Consider the standard action of B on $\mathbb{R}^n$ and let

$$a = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n.$$

Describe $\mathrm{Stab}_B(a)$.

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(5) Consider the elements

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \qquad y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

in the symmetric group S_4 of permuations on four letters. Show that x and y are not conjugate in S_4, i.e. show that there is no element $\sigma \in S_4$ satisfying $y = \sigma x \sigma^{-1}$.

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(6) Let G be a finite group and let $x, y \in G$ be two elements of order two. Show that the subgroup of G generated by x and y is isomorphic to the dihedral group $D_{2|xy|}$.

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(7) Let G be a non-commutative group. Show that $\mathrm{Aut}(G)$ can not be cyclic.

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(8) Suppose a group G acts on a set X. Let $x \neq y \in X$ and let $C = \{g \in G \mid g \cdot x = y\}$. Prove that C is a left coset for $\mathrm{Stab}_G(x)$ and a right coset for $\mathrm{Stab}_G(y)$.

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(9) Let G be a group of order pk with p prime and $1 < k < p$. Prove that G is not simple.

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(10) Let G be a group of order $p_1^2 p_2^2 p_3^2$ with p_1, p_2, and p_3 distinct primes. Suppose that all Sylow subgroups of G are normal. Show that G must be abelian. *Hint:* Show that G must be the product of all it Sylow subgroups.

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(11) Prove that if $|G| = 105$ then G has a normal 5-Sylow subgroup and a normal 7-Sylow subgroup.

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(12) Find the number of p-Sylow subgroups of A_5 for $p = 2, 3, 5$.

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(13) Let $H \lhd G$. Show that H' is a normal subgroup of G.

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(14) Prove that if A and B are solvable, then $A \times B$ is solvable.

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(15) Prove that every group of order n is solvable if (a) $n = 12$, (b) $n = 20$, (c) $n = 100$.

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(16) Prove that the dihedral group D_{16} is nilpotent.
