Math 603. Final Exam
(due by 5pm on Tuesday, May 6, 2008)

1. Suppose G is a finite group, and consider the following data:

(a) the sequence of integers $\{n_d\}_{d \in \mathbb{Z}_{\geq 1}}$, where n_d is the number of irreducible complex representations of G of dimension d;

(b) the group algebra of G.

Can we reconstruct G up to isomorphism from the datum (a)? Can we reconstruct G up to isomorphism from the datum (b)? How will you answers change if in addition we assume that G is abelian?

2. Let p be a prime and let $a \in \mathbb{Q}$ be a rational number which is not the p-th power in $\mathbb{Q}$. Find the Galois group of the polynomial $f(x) = x^p - a$ over $\mathbb{Q}$.

3. Let C be an abelian category and let $A = (A^{\bullet,\bullet}, d_1, d_2)$ be a double complex in C with $A^{i,j} = 0$ for all $i, j < 0$. Assume that all rows and columns of A are exact. Let B be the double complex obtained from A by replacing all $A^{i,0}$ and all $A^{0,j}$ by zero. Show that there is a quasi-isomorphism $A^{0,0} \to \mathrm{tot}(B)$.

4. Let F, G be two presheaves of sets on a topological space X. Define a presheaf H of sets on X by setting $H(U) := \mathrm{Hom}\left(F_{|U}, G_{|U}\right)$ to be the set of morphisms between the presheaves $F_{|U}$ and $G_{|U}$ on U with the natural restriction maps $H(U) \to H(V)$. Show that if F and G are sheaves, then H is a sheaf as well.

5. Suppose $(E_r^{\bullet,\bullet}, d_r)$ is a first quadrant cohomological spectral sequence of complex vector spaces. Suppose $(E_r^{\bullet,\bullet}, d_r)$ converges to a finite dimensional graded vector space $H^\bullet$ equipped with a limiting filtration $F^\bullet H^\bullet$. Suppose $E_2^{p,q} = 0$ unless $p = 0$ or $p = 5$. Show that there exists a long exact sequence of complex vector spaces

$$\cdots \to H^k \to E_2^{0,k} \xrightarrow{d_5} E_2^{5,k-4} \to H^{k+1} \to E_2^{0,k+1} \to \cdots .$$

How is the limiting filtration F related to the maps in this exact sequence?