

Math 603. Final Exam

(due by 4pm on Friday May 8, 2009)

1. Show that up to conjugation $\mathrm{GL}_n(\mathbb{C})$ contains finitely many subgroups of a given finite order.
2. Let K be a field of characteristic $p > 0$.
 - (a) Let $a \in K$. Show that the polynomial $f(x) = x^p - x - a$ is either irreducible or decomposes into a product of linear factors in $K[x]$. Suppose $\lambda \in K$ is a root of f . Describe all the linear factors in the decomposition of $f(x)$.
 - (b) Let $L \supset K$ be a Galois extension of degree p . Show that $L \cong K(\theta)$ where θ is the root of some polynomial of the form $x^p - x - a$ with $a \in K$.
 - (c) Show that $L \supset K$ is a Galois extension with $\mathrm{Gal}(L/K)$ isomorphic to the product of finitely many cyclic groups of order p if and only if $L = K(\theta_1, \dots, \theta_s)$ and θ_i is the root of some polynomial $x^p - x - a_i$ with $a_i \in K$.
3. Let $A \subset B$ be a full subcategory in the additive category B . Suppose that the inclusion functor $i : A \rightarrow B$ has a left adjoint $a : B \rightarrow A$. Show that if $f : x \rightarrow y$ is a morphism in A such that $i(f) : i(x) \rightarrow i(y)$ has a kernel in B , then $a(\ker(i(f)))$ is a kernel for f in A .
4. Let X be a topological space. Suppose that for each $x \in X$ we are given some abelian group A_x . Consider the discrete topological space \mathbf{A} whose underlying set is $\mathbf{A} := \coprod_{x \in X} A_x$. Let $\pi : \mathbf{A} \rightarrow X$ be the natural projection sending $a \in A_x$ to x . This is a continuous map of topological spaces and we can form the sheaf of discontinuous sections \mathcal{A} of π . Explicitly \mathcal{A} is given by

$$U \mapsto \mathcal{A}(U) := \prod_{x \in U} A_x,$$

for all open $U \subset X$.

- (a) Let $\mathrm{Sh}(X, Ab)$ be the category of sheaves of abelian groups on X . Show that if each A_x above is an injective abelian group, then the sheaf $\mathcal{A} \in \mathrm{Sh}(X, Ab)$ is an injective object in the abelian category $\mathrm{Sh}(X, Ab)$.

- (b) Let F be a sheaf of abelian groups on X . Apply the previous construction to the family of abelian groups $\{F_x\}_{x \in X}$, where F_x denotes the stalk of F at x . The resulting sheaf \mathcal{F} is called the sheaf of discontinuous sections of F . Show that there is a natural injective map of sheaves $F \rightarrow \mathcal{F}$ which is the identity on stalks.
- (c) Show that the abelian category $\mathbf{Sh}(X, Ab)$ has enough injectives.
5. Let X be discrete topological space. Let \mathbb{X} be the associated discrete additive category, i.e. $\text{ob}(\mathbb{X}) = X$ and the only morphisms of \mathbb{X} are the identity morphisms.
- (a) Show that there is a natural equivalence of categories between the category $\mathbf{Sh}(X, Ab)$ of sheaves of abelian groups on X and the category $\text{Fun}(\mathbb{X}, Ab)$ of functors between \mathbb{X} and the category Ab of abelian groups.
- (b) Show that the full subcategory of injective objects in $\mathbf{Sh}(X, Ab)$ is equivalent to the category of all functors from \mathbb{X} to the category of injective abelian groups.