Math 603. Final Exam

(due by 5pm on Tuesday May 14, 2019)

1. (a) Check that $x^4 - 2x^2 - 2$ is irreducible over $\mathbb{Q}$ and that it has roots $\pm\sqrt{1 \pm \sqrt{3}}$.

(b) Let $K_1 = \mathbb{Q}(\sqrt{1+\sqrt{3}}) \subset \mathbb{C}$, $K_2 = \mathbb{Q}(\sqrt{1-\sqrt{3}}) \subset \mathbb{C}$. Show that $K_1 \neq K_2$ and that $F := K_1 \cap K_2 = \mathbb{Q}(\sqrt{3})$.

(c) Check that K_1, K_2, and the composite K_1K_2 are Galois over F. Show that $\mathrm{Gal}(K_1K_2|F)$ is isomorphic to the Klein 4-group. Write down the subgroups of $\mathrm{Gal}(K_1K_2|F)$ and determine the corresponding fixed subfields.

(d) Show that the splitting field of $x^4 - 2x^2 - 2$ over $\mathbb{Q}$ is of degree 8 with dihedral Galois group.

2. Let R be a ring and let $f : x \to y$ be a morphism of complexes of R-modules. We say that f is a ***quasi-isomorphism*** if for every $a \in \mathbb{Z}$ the map $H^a(f) : H^a(x) \to H^b(y)$ is an isomorphism of R-modules.

(a) Check that a complex $x \in \mathrm{Compl}(R\text{-mod})$ is acyclic (=exact) if and only if the zero morphism $0 : x \to x$ is a quasi-isomorphism.

(b) Check that a map $f : x \to y$ in $\mathrm{Compl}(R\text{-mod})$ is a quasi-isomorphism if and only if $\mathrm{cone}(f)$ is acyclic.

3. Let $x \in \mathrm{Compl}^b(R\text{-mod})$ be a bounded complex of R-modules. Show that x can be built from objects in $(R\text{-mod}) \subset \mathrm{Compl}^b(R\text{-mod})$ by taking successively cones of morphisms and/or shifts of objects.

4. Let C be an additive category in the sense that: the hom sets of C are equipped with abelian group structures so that the composition is bilinear, C has an initial and a terminal object which coincide, and finite products exist in C.

(a) Let $p_x : x\times y \to x$ and $p_y : x\times y \to y$ be the canoical maps. Define $i_x : x \to x \times y$ and $i_y : y \to x \times y$ to be the maps corresponding to the pairs

$$(\mathrm{id}_x, 0) \in \mathrm{Hom}(x, x) \times \mathrm{Hom}(x, y) = \mathrm{Hom}(x, x \times y),$$
$$(0, \mathrm{id}_y) \in \mathrm{Hom}(y, x) \times \mathrm{Hom}(y, y) = \mathrm{Hom}(y, x \times y)$$

Check that

$$(*) \qquad p_x \circ i_x = \mathrm{id}_x, \quad p_y \circ i_x = 0, \quad p_x \circ i_y = 0, \quad p_y \circ i_y = \mathrm{id}_y,$$

and that

$$(**) \qquad i_x \circ p_x + i_y \circ p_y = \mathrm{id}_{x\times y}\,.$$

Use this to argue that the functors $\mathrm{Hom}(x, -) \times \mathrm{Hom}(y, -)$ and $\mathrm{Hom}(x \times y, -)$ are canonically isomorphic, i.e. that the product $x \times y$ is also a coproduct in C. From now on write $\oplus$ for finite (co)products in C.

(b) Let $z \in \mathrm{ob}(C)$ equipped with maps $p'_x : z \to x$, $p'_y : z \to y$, $i'_x : x \to z$, and $i'_y : y \to z$ satisfying the analogues of conditions $(*)$ and $(**)$. Prove that there exists a unique isomorphism $\varphi : z \to x \times y$ satisfying $p_x \circ \varphi = p'_x$, $p_y \circ \varphi = p'_y$, $\varphi \circ i'_x = i_x$, and $\varphi \circ i'_y = i_y$.

5. Let C be a category that has finite products and coproducts. For any two objects $x, y \in \mathrm{ob}\, C$ we have maps $\Delta_x = \mathrm{id}_x \times \mathrm{id}_x : x \to x \times x$ and $\nabla_y = \mathrm{id}_y \coprod \mathrm{id}_y : y \coprod y \to y$. Also, for any two morphisms $f_1 : x_1 \to y_1$ and $f_2 : x_2 \to y_2$ in C we have the map $f_1 \times f_2 : x_1 \times x_2 \to y_1 \times y_2$ corresponding to the pair $(f_1 \circ p_{x_1}, f_2 \circ p_{x_2})$.

(a) Show that if C is an additive category, and $f, g : x \to y$ are two morphisms in C, then $f + g = \nabla_y \circ (f \times g) \circ \Delta_x$.

(b) Conclude that a category C is additive if and only if: C has an initial and a terminal object that coincide, C has all finite products which are also finite coproducts, the operation on hom sets defined by the formula in part (a) is an abelian group structure for which the composition is bilinear.

6. Suppose R is a ring and let $f : p \to x$ be a map of complexes of R-modules. Suppose that x is acyclic and p is a bounded above complex of projectives. Show that f is homotopic to zero.