Math 603. Homework 1

(due Monday, February 25, 2019)

1. Let A be a commutative ring. Suppose P_1 and P_2 are projective A-modules and M is some A-module. Let

$$0 \longrightarrow K_1 \xrightarrow{i_1} P_1 \xrightarrow{q_1} M \longrightarrow 0$$

$$0 \longrightarrow K_2 \xrightarrow{i_2} P_2 \xrightarrow{q_2} M \longrightarrow 0$$

be short exact sequences of A-modules. The following steps prove the strong version of Schanuel's lemma: $K_1 \oplus P_2$ and $P_1 \oplus K_2$ are isomorphic as A-modules.

(a) Consider the map of A-modules

$$f: \quad P_1 \oplus P_2 \longrightarrow M$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto (q_1, q_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = q_1(x_1) + q_2(x_2)$$

Let $N = \ker(f)$. Show that N fits in a short exact sequence

$$0 \longrightarrow N \longrightarrow P_1 \oplus P_2 \xrightarrow{f} M \longrightarrow 0.$$

(b) Let $\varphi : P_1 \to P_2$ and $\psi : P_2 \to P_1$ be maps of A-modules that fill the projectivity diagrams

$$\begin{array}{ccc} P_1 & & \\ {\scriptstyle\varphi}\downarrow & \searrow^{q_1} & \\ P_2 & \xrightarrow[q_2]{} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} P_2 & & \\ {\scriptstyle\psi}\downarrow & \searrow^{q_2} & \\ P_1 & \xrightarrow[q_1]{} & M \end{array} .$$

Define maps $u, v \in \mathrm{End}_A(P_1 \oplus P_2)$ by the formulas:

$$u = \begin{pmatrix} \mathrm{id}_{P_1} & 0 \\ \varphi & \mathrm{id}_{P_2} \end{pmatrix}, \qquad v = \begin{pmatrix} \mathrm{id}_{P_1} & \psi \\ 0 & \mathrm{id}_{P_2} \end{pmatrix}.$$

Show that u and v are automorphisms and that they induce a commutative diagram of A-modules with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1 \oplus P_2 & \xrightarrow{\begin{pmatrix} i_1 \\ \mathrm{id}_{P_2} \end{pmatrix}} & P_1 \oplus P_2 & \xrightarrow{(q_1,0)} & M & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle v_{|N}} & & \uparrow{\scriptstyle v} & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & P_1 \oplus P_2 & \xrightarrow{f} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u_{|N}} & & \downarrow{\scriptstyle v} & & \| & & \\
0 & \longrightarrow & P_1 \oplus K_2 & \xrightarrow[\begin{pmatrix} \mathrm{id}_{P_1} \\ i_2 \end{pmatrix}]{} & P_1 \oplus P_2 & \xrightarrow[(0,q_2)]{} & M & \longrightarrow & 0
\end{array}$$

(c) Check that $v_{|N} : N \to K_1 \oplus P_2$ amd $u_{|N} : N \to P_1 \oplus K_2$ are isomorphisms.

2. Let A be a commutative ring. Recall that if M, N are A-modules the commutativity of A implies that the abelian group $\mathrm{Hom}_A(M, N)$ of A-module homomorphisms has itself a natural A-module structure given as follows. Given $a \in A$ and an A-module homomorphism $f : M \to N$ we define a new A-module homomorphism

$$a \cdot f : M \to N, \quad x \mapsto af(x),$$

where $af(x)$ denotes the action of a on $f(x)$ in the module N.

Fix a finitely generated projective A-module P.

(a) Write $P^\vee$ for the A-module $\mathrm{Hom}_A(P, A)$. Consider the natural adjunction map $\mathsf{ev} : P \to P^{\vee\vee}$ which to each $x \in P$ assigns the homomorphism $\mathsf{ev}_x : P^\vee \to A$, $\xi \mapsto \xi(x)$ of evaluation on x. Prove that ev is an isomorphism.

(b) Define an A-linear map $\mathrm{tr} : \mathrm{End}_A(P) \to A$ with the property: for any A-linear map $\tau : \mathrm{End}_A(P) \to A$ there exists a unique endomorphism $\varphi \in \mathrm{End}_A(P)$ so that $\tau(f) = \mathrm{tr}(f\varphi)$ for all $f \in \mathrm{End}_A(P)$.

(c) Show that if P is free, then an A-linear map $\tau : \mathrm{End}_A(P) \to A$ satisfies $\tau(f \circ g) = \tau(g \circ f)$ for all $f, g \in \mathrm{End}_A(P)$ if and only if $\tau = a \cdot \mathrm{tr}$ for some $a \in A$.

(d) Given $f \in \mathrm{End}_A(P)$ define $f^\vee \in \mathrm{End}_A(P^\vee)$ as the A-linear map $f^\vee : P^\vee \to P^\vee$, $\xi \mapsto \xi \circ f$. Show that $\mathrm{tr}(f) = \mathrm{tr}(f^\vee)$.

3. (a) Prove that every direct summand in an injective module is injective.

(b) Find an injective resolution of $\mathbb{Z}/p$ as a $\mathbb{Z}$-module.

(c) Let R be an integral domain which is not a field. Suppose M is an R-module which is at the same time injective and projective. Show that $M = \{0\}$.