

Math 603. Homework 1

(due Wednesday, February 11, 2009)

1. Let F be a field and let $\varphi : GL_n(F) \rightarrow \text{End}_F(\text{Mat}_{n \times n}(F))$ be the left regular representation. Explicitly φ is given by $\varphi(A)(X) = AX$.
 - (a) Show that the φ -invariant subspaces of $\text{Mat}_{n \times n}(F)$ are precisely the left ideals in the matrix algebra $\text{Mat}_{n \times n}(F)$.
 - (b) Show that φ is completely reducible.
2. Let F be a field and let $\text{Ad} : GL_n(F) \rightarrow \text{End}_F(\text{Mat}_{n \times n}(F))$ be the adjoint representation. Explicitly Ad is given by $\text{Ad}(A)(X) = AXA^{-1}$.
 - (a) Show that $V_1 = F \cdot I_n$ and $V_2 = \{X \in \text{Mat}_{n \times n}(F) \mid \text{tr}(X) = 0\}$ are Ad -invariant subspaces of $\text{Mat}_{n \times n}(F)$.
 - (b) Show that if $\text{char}(F) \nmid n$ the representation Ad is completely reducible and V_1 and V_2 are its only invariant subspaces.
3. Fix a prime number p . Let $G \subset \mathbb{C}^\times$ be the subgroup of all complex roots of one of order p^n for $n \geq 0$. Prove that the isomorphism classes of irreducible complex representations of G are in a one-to-one correspondence with all sequences $\{a_n\}_{n=1}^\infty$ of integers, satisfying

$$0 \leq a_n \leq p^n, \quad a_n \cong a_{n+1} \pmod{p^n}.$$

4. Let G be a finite p -group. Show that every irreducible representation of G over a field of characteristic p is the trivial one dimensional representation.