Math 603. Homework 2

(due Monday, March 11, 2019)

1. Let A be a commutative ring and let $S \subset A - \{0\}$ be any subset. Write $\mathrm{mult}(S)$ for the multiplicative closure of S and set $S^{-1}A := \mathrm{mult}(S)^{-1}A$. Let $S = \{s_1, \ldots, s_k\} \subset A - \{0\}$. Show that $S^{-1}A = \{s\}^{-1}A$, where $s = s_1 s_2 \ldots s_k$.

2. A multiplicatively closed subset $S \subset A$ in a commutative ring is called *saturated* if $ab \in S$ implies that $a, b \in S$.

(a) Prove that $A^{\times}$, the set of all units in A, is saturated.

(b) If $S \subset A$ is multiplicatively closed, show that there exists a smallest subset $\mathrm{sat}(S) \subset A$, so that $S \subset \mathrm{sat}(S)$ and $\mathrm{sat}(S)$ is saturated. prove that $\mathrm{sat}(S)^{-1}A \cong S^{-1}A$.

(c) Suppose $S \subset A$ is saturated. Show that $A - S$ is a union of prime ideals.

3. Let A be a commutative ring and let $S \subset A$ be a multiplicatively closed subset. Show that the localization functor

$$S^{-1} : A - \mathrm{mod} \to S^{-1}A - \mathrm{mod}$$

preserves the following classes of modules:

(a) free modules,

(b) finitely generated modules,

(c) projecive modules.