

Math 603. Homework 2

(due Monday, February 8, 2010)

1. Let X be a set and let $\rho : X \rightarrow GL(V)$ be a finite dimensional completely reducible representation of X and let $\lambda : X \rightarrow GL(W)$ be a finite dimensional irreducible representation. Suppose

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k \tag{\dagger}$$

is a decomposition of V as a direct sum of minimal ρ -invariant subspaces.

- (a) Show that every subrepresentation of V and every quotient representation of V is isomorphic to a direct sum of some of the (V_i, ρ_{V_i}) 's.
- (b) Suppose that $U \subset V$ is a ρ -invariant subspace such that (U, ρ_U) is isomorphic to (W, λ) . Show that (U, ρ_U) projects non-trivially on some (V_i, ρ_{V_i}) only if (V_i, ρ_{V_i}) is isomorphic to (W, λ) .
- (c) Define the λ -isotypic component $(V_{(\lambda)}, \rho_{(\lambda)})$ as the direct sum of all (V_i, ρ_{V_i}) 's is isomorphic to (W, λ) . Use (b) to argue that $(V_{(\lambda)}, \rho_{(\lambda)})$ is independent of the decomposition (\dagger) .
2. Let S be an affine space over a field k , modelled on the vector space V . Let $f : S \rightarrow S$ be an affine map. Let $Df : V \rightarrow V$ be the unique linear map for which $f(s + v) = f(s) + Df(v)$ for all $v \in V$ and $s \in S$. The map Df is called the *differential* of f . Show that if 1 is not an eigenvalue of Df , then f has a fixed point.
3. Let S be an affine space over \mathbb{R} , and let $M \subset S$ be a convex subset. Show that if $s \notin M$, then the convex hull C of the subset $M \cup \{s\} \subset S$ is given as $C = \cup_{x \in M} \overline{s x}$, where $\overline{s x}$ denotes the convex hull of the points s and x .

4. Let S be a finite dimensional affine space over a field k . Show that the group $\text{Aff}_k(S)$ of affine linear transformations of S admits a faithful representation in $GL_N(k)$ for some appropriately chosen $N > 0$.
5. Let S be a finite dimensional affine space over \mathbb{R} and let M be a convex subset of S . Show that the closure \overline{M} of M is also convex and that every interior point of \overline{M} belongs to M .