

Math 603. Homework 3

(due Monday, February 22, 2010)

1. Let G and H be groups and let $\rho : G \rightarrow GL(V)$, $\lambda : H \rightarrow GL(W)$ be two finite dimensional linear representations over an algebraically closed field. Show that if ρ and λ are irreducible, then

$$\rho \otimes \lambda : G \times H \rightarrow GL(V \otimes W), \quad (\rho \otimes \lambda)(g, h) := \rho(g) \otimes \lambda(h)$$

is also irreducible.

2. Let K be a field of characteristic zero. Let $h(t) \in K[t]$ be a polynomial of degree n . Let $A := K[t]/(h)$.

- (a) Show that A is semi-simple if and only if h has no multiple roots in \overline{K} .
- (b) Let $c_1, \dots, c_n \in \overline{K}$ be all roots of h (counted with multiplicities if necessary). Let $f(t) \in K[t]$, and let $[f] = f(t) + (h)$ be the corresponding element in A . Show that

$$\text{tr}(T_{[f]}) = \sum_i^n f(c_i).$$

- (c) Let $s_i := c_1^i + \dots + c_n^i$ be the power sums of the roots of h . Note that all the s_i can be expressed as polynomials in the coefficients of h and so are in K .

Use the scalar product criterion for semi-simplicity to argue that the polynomial h has no multiple roots in \overline{K} if and only if the matrix

$$\begin{pmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \end{pmatrix}$$

is non-degenerate.

3. Let K be any field. Show that every commutative finite dimensional simple algebra A over K is either a field extension of K or is the one dimensional algebra with zero multiplication.
4. Let A be a finite dimensional semi-simple associative \mathbb{C} -algebra. Let (W, λ) be a finite dimensional complex irreducible representation of A of dimension n . Consider the representation of A given by $(V, \rho) := (W^{\oplus m}, \rho^{\oplus m})$. What is the group of automorphisms of (V, ρ) ?