Math 603. Homework 4
(due Wednesday, April 5, 2019)

1. Let $L \supset K$ be an extension of fields of characteristic $p > 0$. An element $\alpha \in L$ is *separable* over K if α is algebraic over K and the minimal polynomial of α is separable over K. The separable closure K^{sep} of K in L is the set of all elements of L that are separable over K.

(a) Show that K^{sep} is a subfield in L.

(b) If $L \supset K$ is finite, show that there exists an integer $k \geq 0$ so that $L^{p^k} \subset K^{\text{sep}}$.

(c) If $L \supset K$ is algebraic, show that for any $\alpha \in L$ there exists an integer $k \geq 0$ so that $\alpha^{p^k} \in K^{\text{sep}}$.

2. A finite extension $L \supset K$ of fields is called *purely inseparable* if $K^{\text{sep}} = K$. Suppose that we have a tower $K = K_0 \subset K_1 \subset \ldots \subset K_s = L$. Show that L is purely inseparable over K if and only if K_i is purely inseparable over K_{i-1} for all $i = 1, \ldots, s$.

3. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a Galois extension of $\mathbb{Q}$ of degree four. Show that the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{2+\sqrt{2}})|\mathbb{Q})$ is cyclic.