

Math 603. Homework 4

(due Friday, February 22, 2008)

1. Let E be the set of faces of a standard cube, and let V be the 6-dimensional vector space of all \mathbb{C} -valued functions on E . The isomorphism of S_4 with the rotations of the cube gives a permutation action of S_4 on E and so by pullback of functions gives a representation $\rho : S_4 \rightarrow GL(V)$.

(a) Show that the character χ_ρ of (V, ρ) is given by

$$\chi_\rho(g) = \text{the number of faces fixed by } g.$$

(b) Find the decomposition of (V, ρ) into irreducibles.

(c) Describe all minimal ρ -invariant subspaces in V .

2. Let $G \subset S_n$ be a 2-transitive subgroup, i.e. for every $i \neq j, p \neq q \in \{1, \dots, n\}$ there exists $\sigma \in G$, s.t. $\sigma(i) = p, \sigma(j) = q$.

(a) Recall the Burnside formula for the number of orbits of a finite group H acting on a finite set S :

$$\#(S/H) = \frac{1}{\#H} \sum_{h \in H} \#S^h,$$

where $S^h \subset S$ is the subset of h -fixed points on S . Use the Burnside formula to count the number of orbits of G on the set $\{1, \dots, n\} \times \{1, \dots, n\}$.

(b) Show that the representation of G in the space of functions on $\{1, \dots, n\}$ decomposes into a direct sum of two irreducible representations and that one of these is the trivial one dimensional representation.

3. Describe the character table of A_5 .
4. Let G be a finite group and let $(V_1, \rho_1), \dots, (V_s, \rho_s)$ be all complex irreducible representations of G . Suppose (V, ρ) is some finite dimensional representation of G and let $\pi_i : V \rightarrow V$ be the projector of V onto its ρ_i -isotypic component. Show that

(a) The element

$$\frac{\dim V_i}{\dim V} \sum_{g \in G} \chi_{\rho_i}(g^{-1})g \in \mathbb{C}G$$

belongs to $\text{End}(V_i) \subset \mathbb{C}G$ and is equal to id_{V_i} .

(b) Show that

$$\pi_i = \frac{\dim V_i}{\dim V} \sum_{g \in G} \chi_{\rho_i}(g^{-1})\rho(g).$$