

## Math 603. Homework 6

(due Friday, March 7, 2008)

1. Let  $L \supset K$  be an extension of fields of characteristic  $p > 0$ . An element  $\alpha \in L$  is *separable* over  $K$  if  $\alpha$  is algebraic over  $K$  and the minimal polynomial of  $\alpha$  is separable over  $K$ . The separable closure  $K^{\text{sep}}$  of  $K$  in  $L$  is the set of all elements of  $L$  that are separable over  $K$ .
  - (a) Show that  $K^{\text{sep}}$  is a subfield in  $L$ .
  - (b) If  $L \supset K$  is finite, show that there exists an integer  $k \geq 0$  so that the extension  $L^{p^k} \subset K^{\text{sep}}$ .
  - (c) If  $L \supset K$  is algebraic, show that for any  $\alpha \in L$  there exists an integer  $k \geq 0$  so that  $\alpha^{p^k} \in K^{\text{sep}}$ .
2. A finite extension  $L \supset K$  of fields is called *purely inseparable* if  $K^{\text{sep}} = K$ . Suppose that we have a tower  $K = K_0 \subset K_1 \subset \dots \subset K_s = L$ . Show that  $L$  is purely inseparable over  $K$  if and only if  $K_i$  is purely inseparable over  $K_{i-1}$  for all  $i = 1, \dots, s$ .
3. Show that  $\mathbb{Q}(\sqrt{2 + \sqrt{2}})$  is a Galois extension of  $\mathbb{Q}$  of degree four. Show that the Galois group  $\text{Gal}(\mathbb{Q}(\sqrt{2 + \sqrt{2}})|\mathbb{Q})$  is cyclic.