

Math 603. Homework 6

(due Friday, April 8, 2009)

1. Let $\mathbb{A} = (A^{\bullet,\bullet}, \partial, \delta)$ be a cohomological double complex of abelian groups. Thus $\partial : A^{\bullet,\bullet} \rightarrow A^{\bullet+1,\bullet}$, $\delta : A^{\bullet,\bullet} \rightarrow A^{\bullet,\bullet+1}$, and $\partial^2 = 0$, $\delta^2 = 0$, $\delta\partial = \partial\delta$. Consider the (coproduct) total complex $\text{tot}(\mathbb{A}) = (\text{tot}(\mathbb{A})^\bullet, d_{\text{tot}(\mathbb{A})}^\bullet)$, defined by

$$\text{tot}(\mathbb{A})^n := \bigoplus_{p+q=n} A^{p,q}, \quad d_{\text{tot}(\mathbb{A})}^n = \partial + (-1)^p \delta \text{ on } A^{p,q}.$$

- (a) Check that $\text{tot}(\mathbb{A})$ is a complex, i.e. check that $d_{\text{tot}(\mathbb{A})} \circ d_{\text{tot}(\mathbb{A})} = 0$.
- (b) Suppose that \mathbb{A} is bounded, i.e. suppose that \mathbb{A} has at most finitely many non-zero groups along each diagonal line $n = p + q$. Show that if all the rows of \mathbb{A} are exact, then $\text{tot}(\mathbb{A})$ is acyclic.
2. Let C be a complex of abelian groups and let $Z(C)$, $B(C)$, and $H(C)$ be the complexes of cocycles, coboundaries, and cohomology of C (with zero differentials). Show that there are short exact sequences of complexes

$$0 \longrightarrow Z(C) \longrightarrow C \xrightarrow{d_C} B(C)[1] \longrightarrow 0$$

and

$$0 \longrightarrow H(C) \longrightarrow C/B(C) \xrightarrow{d_C} Z(C)[1] \longrightarrow H(C)[1] \longrightarrow 0.$$

3. Let $f : M \rightarrow N$ be a map of complexes of abelian groups.
- (a) Show that the cone complex $\text{cone}(f)$ fits in a short exact sequence of complexes

$$0 \rightarrow M \rightarrow \text{cone}(f) \rightarrow N[1] \rightarrow 0.$$

- (b) Let $N \hookrightarrow C$ be an embedding of complexes and suppose that for all n , the map of abelian groups $N^n \hookrightarrow C^n$ is split, i.e. for each n we can choose an isomorphism

$$\varphi^n : C^n \xrightarrow{\cong} N^n \oplus (C/N)^n$$

compatible with the embedding of $N^n \subset C^n$ (but not necessarily compatible with the differentials). Set $M := (C/N)[-1]$ and use the splittings φ^n to write the differential d_C in a block form

$$d_C = \begin{pmatrix} d_N & f \\ 0 & d_{M[1]} \end{pmatrix}.$$

Show that $f : M \rightarrow N$ is a morphism of complexes and that the splitting isomorphisms $\varphi = (\varphi^\bullet)$ identify C with $\text{cone}(f)$.