Math 603. Homework 7
(due Wednesday, April 22, 2009)

Let $\mathcal{A}$ be an abelian category. Fix $x, y \in \text{ob}(\mathcal{A})$. Define the *set of extensions* $E(x, y)$ of x by y to be the set of equivalence classes of short exact sequences

$$\xi: \ 0 \longrightarrow y \xrightarrow{u} z \xrightarrow{v} x \longrightarrow 0,$$

where $\xi \cong \xi'$ if and only if we can find a map $\alpha : z \to z'$ which fits in a commutative diagram

$$\begin{array}{ccccccccccc} \xi: & 0 & \longrightarrow & y & \xrightarrow{u} & z & \xrightarrow{v} & x & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle\alpha} & & \| & & \\ \xi': & 0 & \longrightarrow & y & \xrightarrow{u'} & z' & \xrightarrow{v'} & x & \longrightarrow & 0. \end{array}$$

(a) Given a morphism $\varphi : y \to y'$ and an extension $\xi \in E(x, y)$, define $\varphi \circ \xi \in E(x, y')$ as the class of the short exact sequence

$$0 \longrightarrow y' \longrightarrow y' \textstyle\coprod_y z \longrightarrow x \longrightarrow 0.$$

Check that this is well defined. Show that this operation respects composition, that is for every $\varphi : y \to y'$, and $\psi : y' \to y''$, and every $\xi \in E(x, y)$, we have $\psi \circ (\varphi \circ \xi) = (\psi\varphi) \circ \xi$.

(b) Given $\varphi : x' \to x$ and an extension $\xi \in E(x, y)$, define $\xi \circ \varphi \in E(x', y)$ as the class of the short exact sequence

$$0 \longrightarrow y \longrightarrow x' \times_x z \longrightarrow x \longrightarrow 0.$$

Check that this is well defined. Show that for every $\psi : x'' \to x'$ we have $(\xi \circ \varphi) \circ \psi = \xi \circ (\varphi\psi)$. Show that if $\mu : y \to y'$ and $\nu : x' \to x$, then $\mu \circ (\xi \circ \nu) = (\mu \circ \xi) \circ \nu$.

(c) Given $x, y \in \mathrm{ob}(\mathcal{A})$, let $\Delta_x : x \to x \oplus x$ be the diagonal morphism and let $\nabla_y : y \oplus y \to y$ be the codiagonal morphism. Given $\xi, \xi' \in E(x, y)$ consider $\xi \oplus \xi' \in E(x \oplus x, y \oplus y)$, defined as the class of the short exact sequence

$$0 \longrightarrow x \oplus x \longrightarrow z \oplus z \longrightarrow y \oplus y \longrightarrow 0.$$

Define the *Baer sum* of ξ and ξ' by setting $\xi + \xi' := \nabla_y \circ (\xi \oplus \xi') \circ \Delta_x$. Show that operation turns $E(x, y)$ into abelian group. What is the identity element in this group?

(d) Let $\cdots p^{-2} \to p^{-1} \to p^0 \to x \to 0$ be a projective resolution of x. Show that $E(x, y)$ is isomorphic to the first cohomology group of the complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(p^0, y) & \to & \mathrm{Hom}(p^{-1}, y) & \to & \mathrm{Hom}(p^{-2}, y) & \to & \cdots \\ -1 & & 0 & & 1 & & 2 & & \end{array}$$

How would you compute $E(x, y)$ by using an injective resolution of y?