Math 603. Homework 8

(due Friday, April 11, 2008)

Let $X : \mathbb{\Delta}^{\mathrm{op}} \to (\mathsf{Sets})$ be a simplicial set. Consider the following standard maps in $\mathbb{\Delta}$:

the i-th face: $\partial_n^i : [n-1] \to [n]$ - the strictly increasing injection whose image does not contain $i \in [n]$.

the i-th degeneracy: $\sigma_n^i : [n+1] \to [n]$ - the non-decreasing injection which takes the value $i \in [n]$ twice.

1. Show that:

- every strictly increasing injective map in $\mathbb{\Delta}$ can be decomposed into a chain of face maps.
- every surjective map in $\mathbb{\Delta}$ can be decomposed into a chain of degeneracy maps.
- every map in $\mathbb{\Delta}$ can be decomposed into a chain of maps which are either chain maps or degeneracy map.

2. Show that the face and degeneracy maps satisfy the following relations:

$$\partial_{n+1}^j \partial_n^i = \partial_{n+1}^i \partial_n^{j-1} \qquad i < j$$
$$\sigma_n^j \sigma_{n+1}^i = \sigma_n^i \sigma_{n+1}^{j+1} \qquad i \leq j$$

$$\sigma_{n-1}^j \partial_n^i = \begin{cases} \partial_{n-1}^i \sigma_{n-2}^{j-1} & i < j; \\ \mathrm{id}_{[n-1]} & i = j \text{ or } i = j+1; \\ \partial_{n-1}^{i-1} \sigma_{n-2}^j & i > j+1; \end{cases}$$

3. Suppose that $f : [m] \to [n]$ is a morphism in $\mathbb{\Delta}$. Show that f can be written uniquely as

$$f = \partial_n^{i_1} \partial_{n-1}^{i_2} \cdots \partial_{n-s+1}^{i_s} \sigma_{m-t}^{j_t} \cdots \sigma_{m-2}^{j_2} \sigma_{m-1}^{j_1},$$

where $n \geq i_1 > i_2 > \cdots > i_s \geq 0$ and $m \geq j_1 > j_2 > \cdots > j_t \geq 0$, $n = m - t + s$. Use this to show that the relations in **(2)** generate all the relations amon face and degeneracy maps.