Math 603. Homework 1

Hints on problem 2

2. Let A be a commutative ring. Recall that if M, N are A-modules the commutativity of A implies that the abelian group $\mathrm{Hom}_A(M, N)$ of A-module homomorphisms has itself a natural A-module structure given as follows. Given $a \in A$ and an A-module homomorphism $f : M \to N$ we define a new A-module homomorphism

$$a \cdot f : M \to N, \quad x \mapsto af(x),$$

where $af(x)$ denotes the action of a on $f(x)$ in the module N.

Fix a finitely generated projective A-module P.

(a) Write $P^\vee$ for the A-module $\mathrm{Hom}_A(P, A)$. Consider the natural adjunction map $\mathsf{ev} : P \to P^{\vee\vee}$ which to each $x \in P$ assigns the homomorphism $\mathsf{ev}_x : P^\vee \to A$, $\xi \mapsto \xi(x)$ of evaluation on x. Prove that ev is an isomorphism.

(b) Define an A-linear map $\mathrm{tr} : \mathrm{End}_A(P) \to A$ with the property: for any A-linear map $\tau : \mathrm{End}_A(P) \to A$ there exists a unique endomorphism $\varphi \in \mathrm{End}_A(P)$ so that $\tau(f) = \mathrm{tr}(f\varphi)$ for all $f \in \mathrm{End}_A(P)$.

(c) Show that an A-linear map $\tau : \mathrm{End}_A(P) \to A$ satisfies $\tau(f \circ g) = \tau(g \circ f)$ for all $f, g \in \mathrm{End}_A(P)$ if and only if $\tau = a \cdot \mathrm{tr}$ for some $a \in A$.

(d) Given $f \in \mathrm{End}_A(P)$ define $f^\vee \in \mathrm{End}_A(P^\vee)$ as the A-linear map $f^\vee : P^\vee \to P^\vee$, $\xi \mapsto \xi \circ f$. Show that $\mathrm{tr}(f) = \mathrm{tr}(f^\vee)$.

Some hints:

For part (b) use to identification $\mathrm{End}_A(P) \cong P^\vee \otimes_A P$ to define tr. Argue that for every decomposable tensor $\xi \times x$ with $\xi \in P^\vee$ and $x \in X$ we will have to have $\mathrm{tr}(\xi \otimes x) = \xi(x)$.

Part (c) turns out to be more complicated than I originally intended so in the homework I have changed the statement to the stronger assumption that P is not just finitely generated projective but is actually a free A-module of finite rank.

Here is how you can go on aboout proving the statement in general.

Let P be projective of finite rank, and let $\tau : P \to A$ be an A-linear map such that $\tau(fg) = \tau(gf)$ for all endomorphisms f and g. As we know for every A-linear τ there is a unique endomorphism φ so that $\tau(f) = \mathrm{tr}(f\varphi)$ for all f.

If g corresponds to the decomposable tensor $\xi \otimes x$, then $g\varphi$ corresponds to the decomposable tensor $(\xi \circ \varphi) \otimes x$ while φg corresponds to the decomposable tensor $\xi \otimes \varphi(x)$. Thus we compute that

$$\tau(fg) = \mathrm{tr}(fg\varphi) = \xi(\varphi(f(x))) \quad \text{and} \quad \tau(gf) = \mathrm{tr}(gf\varphi) = \xi(f(\varphi(x))),$$

or equivalently

$$\xi([f, \varphi](x)) = 0$$

for all ξ and x. Thus $\mathrm{ev}_{[f,g](x)} : P^\vee \to A$ is the zero map and so $[f, g](x) = 0$ for all x since the adjunction $P \to P^{\vee\vee}$ is an isomorphism. This shows that φ commutes with every endomorphism $f : P \to P$.

Now let $Z \subset \mathrm{End}_A(P)$ be the center of the algebra $\mathrm{End}_A(P)$, i.e. the submodule of all endomorphisms that commute with every other endomorphism. Let $S = \{a \cdot \mathrm{id} \mid a \in A\}$ be the submodule of scalar endomorphisms.

We have natural inclusions of submodules $S \subset Z \subset \mathrm{End}_A(P)$ and we want to argue that φ belongs to S.

Suppose first that the module P is free and let $x_1, \ldots, x_n$ be a system of generators. Then every endomorphism corresponds to an $n \times n$ matrix with entries in A and the trace map is given by the usual matrix formula.

If n equals 1 the statement is obvious. Suppose that $n > 1$. We want to show that if $\varphi \in \mathrm{Mat}_{n \times n}(A)$ commutes with every other matrix, then φ must be a scalar matrix. This is the usual linear algebra argument. If $i \neq j$ and E_{ij} is the elementary matrix with entry 1 at place (i, j) and entry zero everywhere else, then the requirement that φ commutes with $I_n + E_{ij}$ for all j says that the (i,i) diagonal entry is the only non-zero entry in the i-th row of φ. This shows that the matrix φ is diagonal. Next let s_{ij} be the matrix that switches x_i and x_j and keeps all remaining generators fixed. Requiring that φ commutes with s_{ij} implies that the i-th and j-th diagonal entries of φ are equal.

For the general case of P projective you can use a local to global argument and notice that for all prime ideal $\mathfrak{p}$ the localized module $P_{\mathfrak{p}}$ is free and so one can use the above argument to show that $\varphi_{\mathfrak{p}} = 0 \in (Z/S)_{\mathfrak{p}}$.