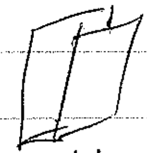
1-1. ①

Problem :

Resolve the singularity $\mathbb{C}^2$/ Binary Icosahedral group.

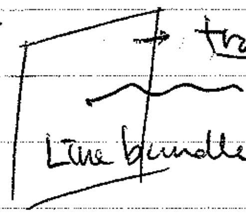
Physics D-branes

Boundary cond. on open string



U(2)-bundle.

10 dim. spacetime



→ transverse Fields

deg of freedom asso. to moving D-brane.

Line bundle

B-field. - Intrinsic 2-form assoc. to strings.
(Gerbe connection)

For open strings - 1-form at end of String
(bundle connection)

Try to describe physics of D-brane.

Try to make effective D-brane "Action"
Dirac-Born-Infeld DBI-action.

Go to a limit of DBI action to decouple gravity
(Maldacena)

U(N)-gauge theory (w/ no gravity) $(N \to \infty)$

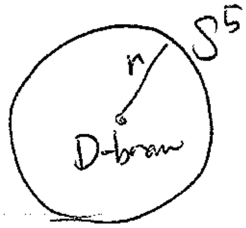
E.g D3-brane :(3-Space and 1-time direction).

$\begin{pmatrix} \text{4-dim'l space} \\ \text{within D-brane} \end{pmatrix} \times \begin{pmatrix} \text{6-dim'l} \\ \text{Black hole} \end{pmatrix}$ real product (product metric)

not real product

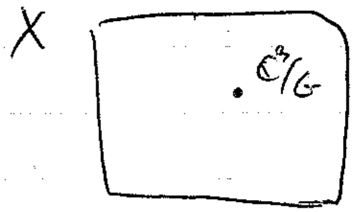
$\left[\begin{pmatrix} \text{4 dim space} \\ \text{with D-Brane} \end{pmatrix} \times \begin{pmatrix} \text{radial Direction} \end{pmatrix} \right] \times$ (5-Sphere).

$AdS_5 \times S^5$ ← sphere around D-brane



Space time is $M_4 \times (\text{six dim})^{\kappa}$ X.

Suppose we have a singular 6-dim'l space.



D-Brane has "horison" S^5/G.

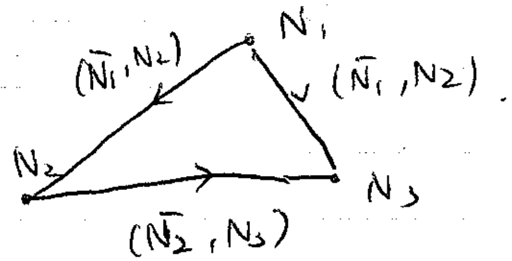
<u>Douglas-Moore</u>

$AdS_5 \times \underbrace{S^5}_{\text{horison}}$.

D-brane action is a Quiver Gauge theory

$U(N_1) \times U(N_2) \times U(N_3) \times \cdots$

with chiral field in reps $(\bar{N}_1, N_2)$ $(\bar{N}_2, N_3)$ $(\bar{N}_1, N_3)$ ⋯



$U(N)$ $\quad N \quad \bar{N}$ $\qquad U(N_1) \times U(N_2)$ $\quad (\bar{N}_1, N_2)$.

N: N-dim rep; $\bar{N}$: complex conj.

This Quiver is the Mckay Quiver for G

<u>Problem</u> Given any singularity, what is quiver?

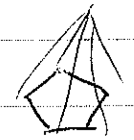
Quiver is an Ext-quiver.

Mckay Correspondence

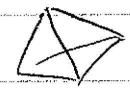
Mckay (1980)

f.s.gp. $G \subset SL(2,\mathbb{C})$
$\subset SU(2)$

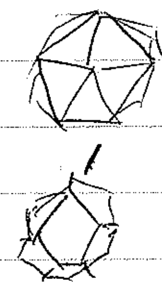
$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1$$

A_n  $\mathbb{Z}_5$

D_n  Dihedral groups

 E_6

 E_7

 E_8.

$G \subset SL(2\,\mathbb{C})$ is a finite gp.

Let Q be the 2-dim rep of G.

Let ρ_i be the irreps of G.

Define n_{ij} by $Q \otimes \rho_i = \bigoplus_j \rho_j^{\oplus n_{ij}}$.

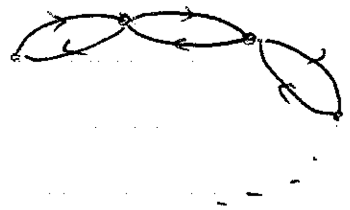
Quiver has a node for each irrep. n_{ij} arrows from node i to node j.

Let's do $\mathbb{Z}_n$. $Q^{(g)} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ $\quad \alpha = e^{2\pi i/n}$. mod n.

All irreps are 1-dim. Let g gen. $\mathbb{Z}_n$. $\quad (\rho_{i+1} \oplus \rho_{i-1})(g^k)$

$\rho_i(g) = \alpha^i \qquad i = 0, \dots, n-1$ $\qquad \|$

$Q(g^k) = \begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^{-k} \end{pmatrix}$. $\qquad (Q \otimes \rho_i)(g^k) = \begin{pmatrix} \alpha^{(i+1)k} & 0 \\ 0 & \alpha^{(i-1)k} \end{pmatrix}$



n node.

Hypersurface singularities

$\mathbb{Z}_n$ g acts on $\mathbb{C}^2$ with coords s and t.

$$g(s,t) = (\alpha s, \alpha^{-1} t)$$

what is $\mathbb{C}[s,t]^G$. $\quad \underset{x}{\overset{\shortparallel}{s^n}}, \underset{z}{\overset{\shortparallel}{st}}, \underset{y}{\overset{\shortparallel}{t^n}}$

$$k[s,t]^G = \frac{k[x,y,z]}{(xy - z^n)}$$

This is a statement in alg geom.

$\mathbb{C}^2/\mathbb{Z}_n$ is the hypersurface $xy - z^n$ in $\mathbb{C}^3$.

$W = xy - z^n$. $\quad \frac{\partial W}{\partial x} = \frac{\partial W}{\partial y} = \frac{\partial W}{\partial z} = W = 0$ at $x = y = z = 0$

Binary icosahedral Group

$$I_K(s,t) = s^{10}t - 11 s^6 t^6 - s t^{11}$$

$$D_K(s,t) = s^{20} + 228 s^{15} t^5 + \cdots$$

$$T_K(s,t) \cdots$$

Quotient is $x^2 + y^3 + z^5 = 0$ in $\mathbb{C}^3$ Klein

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$\mathbb{P}^n$ $[x_0 \cdots x_n]$ $\qquad W_n \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ $\qquad \forall i,j.$ $x_i y_j = x_j y_i$.

$[x_0 \cdots x_n]$ $(y_0, \cdots, y_n)$

Fix a point in $\mathbb{P}^n$, then $(y_0, \cdots, y_n)$ determed up to multiple.

$\rightsquigarrow$ W_n is a cpx line bundle over $\mathbb{P}^n$.

Tautological line bundle.

Put patches on $\mathbb{P}^n$.

$U_j : \{x_j \neq 0\}$ give affine patch $\left(\frac{x_0}{x_j}, \cdots, \frac{x_n}{x_j}\right)$

$y_i = \frac{x_i}{x_j} y_j$, so y_j determines other y's

$\Rightarrow$ y_j is a good coord on fiber.

Any line bundle on $\mathbb{P}^n$ is equi with transition fcns.

$y_i = \left(\frac{x_j}{x_i}\right)^d y_j$ for some $d \in \mathbb{Z}$. $\rightsquigarrow \mathcal{O}(d)$.

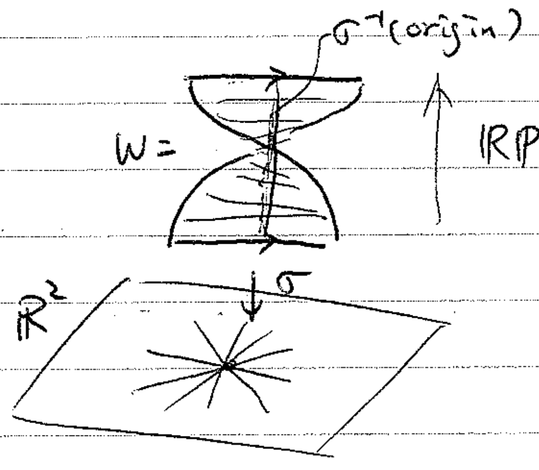
$W_n = \mathcal{O}(-1)$.

Consider $\sigma : W_n \xrightarrow{y_i \text{ coord}} \mathbb{C}^{n+1}$ $\qquad \sigma^{-1}(p)$ is a pt. if $p \neq$ origin

$\sigma^{-1}(\text{origin}) = \mathbb{P}^n$.

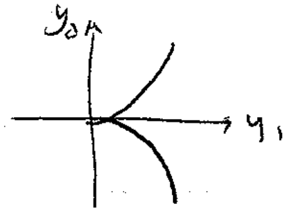
Real case $\overset{S^1}{\shortparallel}$

$W \subset \mathbb{RP}^1 \times \mathbb{R}^2$



Consider drawing a singular curve in $\mathbb{R}^2$

e.g $y_0^2 - y_1^3 = 0$



In patch $U_1 = (x_1 \neq 0)$ $\quad x_1 y_0 = x_0 y_1$

Good affine coord. $\dfrac{x_0}{x_1} = \dfrac{y_0}{y_1}$ and y_1

Let $y_0' = \dfrac{y_0}{y_1}$ $\quad y_1' = y_1$

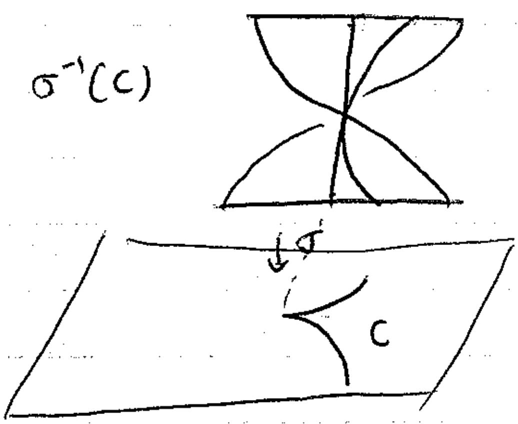
i.e $y_0 = y_0' y_1'$ $\quad y_1 = y_1'$ is tranformation from old to new coord.

σ

$y_0^2 + y_1^3 = 0$ becomes

$y_1'^2 (y_0'^2 + y_1') = 0$. $\quad$ exceptional set is $y_1' = 0$.

$\sigma^{-1}(C)$



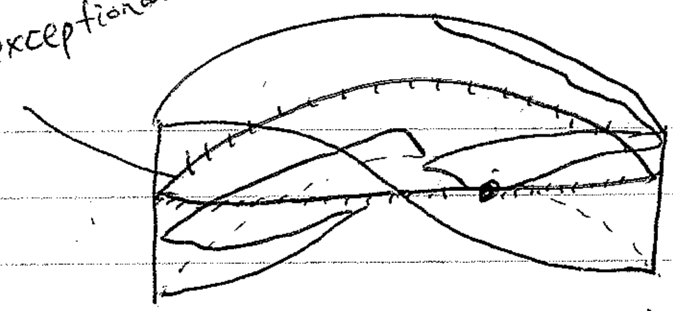
Note If $\sigma : X \to Y$ is a map bet'n var.

then $\exists$ a induced ring map $A(Y) \to A(X)$.

$y_0 = y_1' y_0'$ $\quad y_1 = y_1'$ is exactly this map.

Other path $\quad y_0 = y_1'$ $\quad y_1 = y_0' y_1'$

$\leadsto$ $y_0'^2 (1 + y_0' y_1'^3) = 0$. $\quad$ doesn't pass thru O smooth.



$\sigma^{-1}(C) = E \cup C'$

E = exceptional set.

C' is "proper transform" of C.

<u>Def</u> C' is the (Zariski) closure of $\sigma^{-1}(C \setminus \{0\})$

C' is smooth!

E.g. $\mathbb{C}^2/\mathbb{Z}_n$ $\quad xy = z^n$.

change of coord $\quad X^2+Y^2+Z^n=0$

$n=2$ $\quad X^2+Y^2+Z^2=0$

Let U_x be the patch when $x \neq 0$.

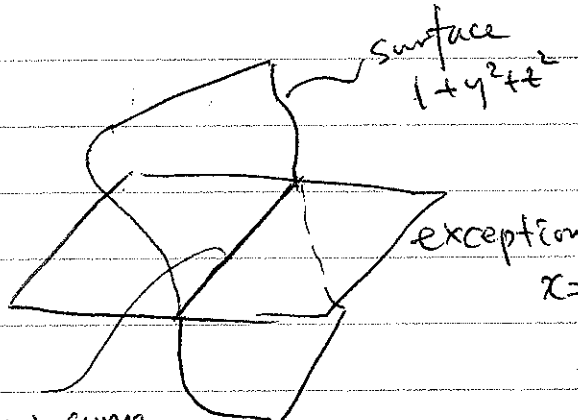
$x \to x'$

$y \to x'y'$ $\qquad x^2+y^2+z^2 \to x^2(1+y^2+z^2)$

$z \to x'z'$ $\qquad X=0$ is exceptional set.

$1+y^2+z^2$ is proper transf.

sm.



C is an exceptional curve within <u>surface</u>

C

$(x, 1+y^2+z^2)$ in patch U_x

Similarly (y, x^2+1+z^2) in patch U_y

(z, x^2+y^2+1) -- U_z

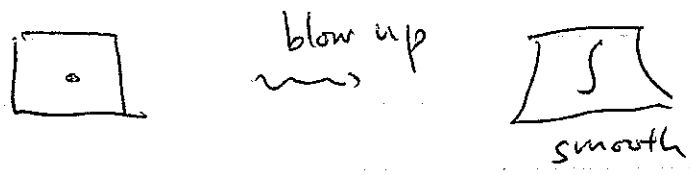
Exceptional curve is $x_0^2+x_1^2+x_2^2=0$ for homog. coord in $\mathbb{P}^2$

This is a $\mathbb{P}^1$: $[a,b] \longrightarrow [a^2, ab, b^2]$

$x_0x_2 = x_1^2$

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$.

Shown : $\mathbb{C}^2/\mathbb{Z}_2$ can be resolved with exceptional $\mathbb{P}^1$



$\mathbb{C}^2/\mathbb{Z}_3$ $\quad x^2+y^2+z^3=0$.

U_x $\quad x^2(1+y^2+\cancel{x}z^3)=0$.

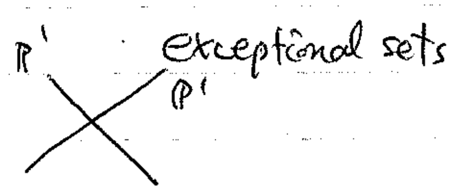
$U_z \rightsquigarrow z^2(x^2+y^2+z)=0$.

In patch U_z, we have for exceptional curve $x^2+y^2=0$

Ie. $x_0^2+x_1^2=0$ in $\mathbb{P}^2$.

$\|$

$(x_0+ix_1)(x_0-ix_1)=0$



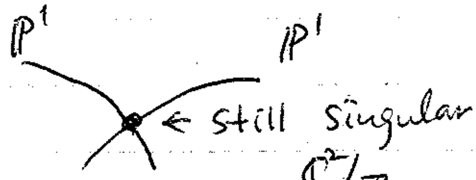
$\mathbb{C}^2/\mathbb{Z}_4$ $\quad x^2+y^2+z^4=0$. — smooth.

U_x $\quad x^2(1+y^2+x^2z^4)$ — $\mathbb{C}^2/\mathbb{Z}_2$

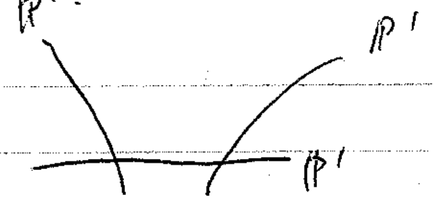
U_z $\quad z^2(x^2+y^2+z^2)$.

exceptional curve same as $\mathbb{C}^2/\mathbb{Z}_3$

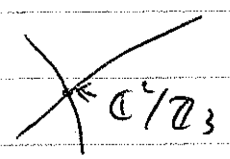
$\mathbb{C}^2/\mathbb{Z}_4$ after one blow up is

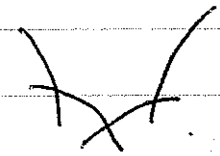


So blow up again

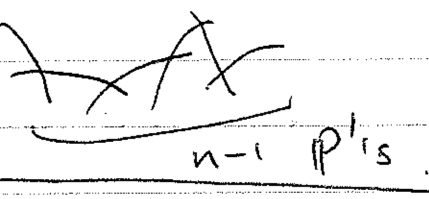


$\mathbb{C}^2/\mathbb{Z}_5$. after one blowup





$\mathbb{C}^2/\mathbb{Z}_n$



HW $\mathbb{C}^2/\mathbb{Z}_2$ resolved is a line bundle over $\mathbb{P}^1$

$\mathcal{O}(d)$ what is d? $(d=-2)$

Also do $x^2+y^3+z^5$

Multiplicity.

Let $\mathfrak{m}$ denote the maximal ideal of the origin in $\mathbb{C}^2/G$.

i.e. for $x^2+y^3+z^5$. it is (x, y, z).

Let $\sigma^*\mathfrak{m}$ denote the pullback to the resolution.

Let R be the coord ring of resolution.

There is a natural map $i : \sigma^*\mathfrak{m} \longrightarrow R$.

Let M be the cokernel of this map.

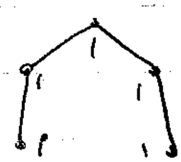
M is a sum of sheaves $\mathcal{E}_i$ each supported on an exceptional curve C_i

Define $l_i = \text{Length}(\mathcal{E}_i)$

where length is the maximal length at a chain of ~~sub~~sheaves $\cdots \subset \mathcal{E}_i'' \subset \mathcal{E}_i' \subset \mathcal{E}_i$ each $\mathcal{E}_i''$ supported on C_i

Any resolution of $\mathbb{C}^2/G$ is associated to a directionless quiver Each node $\mathbb{P}^1$

Join node by a line if $\mathbb{P}^1$'s intersect.

$\mathbb{C}^2/\mathbb{Z}_6$ 

6 Bridgeland stability

7. ~~2.~~ Stability for conifolds and $\mathbb{C}^3/\mathbb{Z}_3$

2. Quiver for $\mathbb{C}^2/D_4$ with relations ~~from~~ tilting module

3. Cohomology of Toric sheafes (Eisenbud, Mustata,

1. Serre's construction of sheaves on $\mathbb{P}^n$. (Hartshorne Ex II.5.9)

HW Fractional Branes on $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_5$

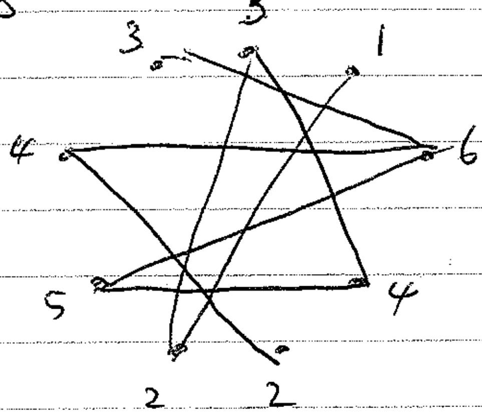
4. del Pezzo surfaces and exceptional collections

5 Tilting Equivalence

2-1

6/16/2006. 1st.

E8



Mckay quiver

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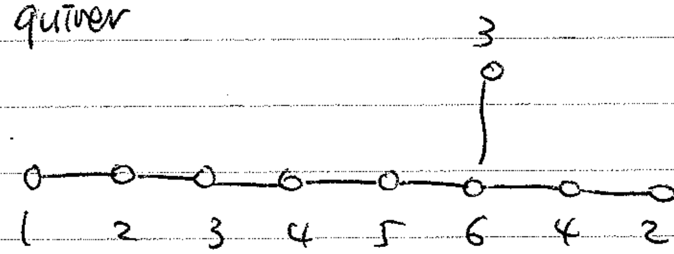


$x^2+y^3+z^5$

$x^2(1+xy^3+x^3z^5)=0$.

$y^2(x^2+y+y^3z^5)$

$z^2\boxed{(x^2+zy^3+z^3)}$

exceptional curve $\mathbb{P}^1$.

$\mathfrak{m}=(x,y,z)$

$\to (xz, yz, z)=(z)$

$(z)+(x^2+zy^3+z^3)=(z,x^2)$.

$\downarrow$ blow up

$x^2(1+\cdots)$

$y^2\boxed{(x^2+zy^2+yz^3)}$

$z^2(x^2+z^2y^3+z)$

length 2.

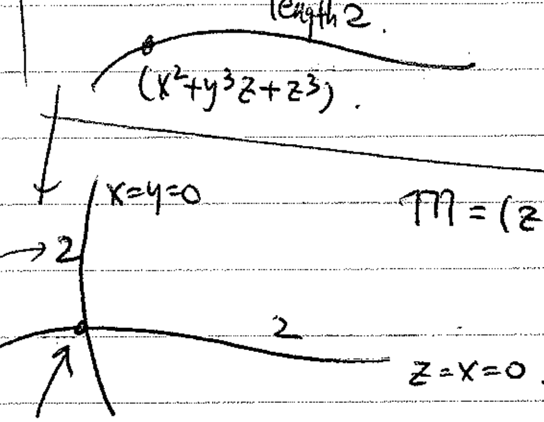
$(x^2+y^3z+z^3)$.

$\mathfrak{m}=(zy)$.

$(zy)+(x^2+y^2z+z^3y)$

$=(x^2,yz)$

$=(x^2,y)\cap(x^2,z)$



Singularity. $x^2+zy^2+yz^3$.

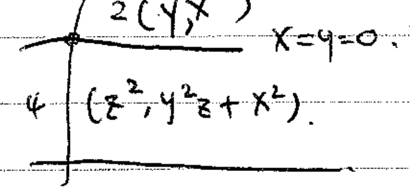
$\downarrow$ blow up

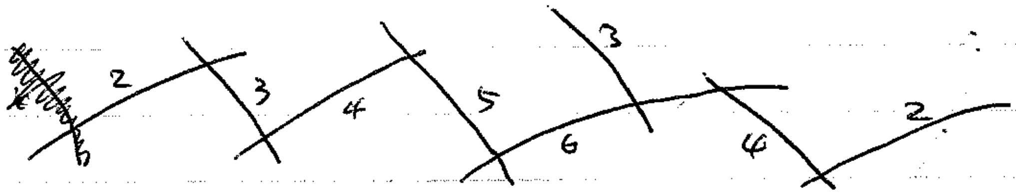
U_z $\quad z^2(x^2+y^2z+z^2y)$ $\quad \mathfrak{m}=(yz^2)$ $\quad \mathfrak{m}+(x^2+y^2z+z^3y)$

$=(yz^2, x^2+y^2z)$

$=(z^2, y^2z+x^2)\cap(x^2,y)$.

U_y





G-equivariant sheaves (coherent sheaf)

Coherent sheaves on $\mathbb{C}^2$ $S = k[s,t]$

A coherent sheaf is a $k[s,t]$-module

i.e. a k-v.s. w/ an action of s and t.
s and t commute

E.g. $M = S$ structure sheaf $\sim$ trivial l.b. on $\mathbb{C}^2$.

E.g. $M = k$. s, t annihilate M.
$= \text{Sky}$: Scraper sheaf at 0.

Put $\mathfrak{m} = (s-a, t-b)$ $a, b \in k$. $(a,b) \neq (0,0)$

localization $M_{\mathfrak{m}} = 0$

Let $G \subset SL(2,\mathbb{C})$ act on s and t.

we have a natural group

$k^2 \rtimes G$. $(t, g)(t', g') = (t \cdot g(t'), gg')$.

translation, rotation

So, a sheaf w/ a G-action is thought of as an "$S * G$"-module where $S * G$ is crossed algebra with above product

"G-equivariant sheaf"

An $S*G$ module is a v.s. and rep of G.

$M = \bigoplus_i V_i \otimes_k \rho_i$ $\quad$ ρ_i are irreps of G.
V_i is a G-inv. v.s.

Let $Q = k^2$ – the 2-dim'l rep of G.
s and t are coords on Q, i.e. elts of Q^*.

The action of s and t on M is given by

$$
\begin{aligned}
y \in & \operatorname{Hom}_G(Q^*, \underset{M^* \otimes M}{\underset{\shortparallel}{\operatorname{End}(M)}}) \\
&= \operatorname{Hom}_G(M, Q \otimes M) \\
&= \operatorname{Hom}_G\Big(\bigoplus_i V_i \otimes_k \rho_i, \bigoplus_j V_j \otimes (Q \otimes \rho_j)\Big) \\
&= \operatorname{Hom}_G\Big(\bigoplus_i V_i \otimes \rho_i, \bigoplus_{j,k} V_j \otimes \rho_k^{\oplus n_{jk}}\Big)
\end{aligned}
$$

Schur's lemma $\Rightarrow = \bigoplus_{i,j} \operatorname{Hom}(V_i, V_j)^{\oplus n_{ji}}$

y is a "representation" of the McKay Quiver

Given a quiver, a rep. associates a v.s. V_i to each node of dim d, and a $d_j \times d_i$ matrix for each arrow $i \to j$.

Converse? need a commutativity of actions of s and t.

Morita Equivalence

Let $R . S$ be rings

R-mod , S-mod : ctg of f.g modules

Thm TFAE.

(1) The ctgs R-mod & S-mod are equivalent.

(2) S-mod has an object P which is a generator and $R \cong \mathrm{End}_S(P)$

(3) There exists an (S,R)-mod. M. s.t. $M \otimes_R -$: R-mod $\longrightarrow$ S-mod. is an equivalence

An object P is a generator if every obj in the ctg is an epimorphic image of a sum of copies of P. and P is projective.

HW S is a generator for the ctg of $(S*G)$mod.

Let $R = Z(S*G)$.

Consists of elts $(\overset{k[s,t]}{p}, \overset{G}{g})$ where $\underbrace{\begin{matrix} p \in k[s,t]^G \\ g = 1. \end{matrix}}_{HW}$

$R = k[s,t]^G$.

Let $S = \bigoplus_i M_i \otimes \rho_i$ as an $(S*G)$-mod.

Then M_i are R-modules

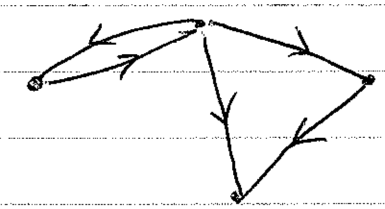
By Schur's lemma.

$$\underline{\mathrm{End}_{S*G}(S) = \mathrm{End}_R(\bigoplus_i M_i)}.$$

Let $A = \mathrm{End}_R(\bigoplus_i M_i)$.

Morita $\Rightarrow$ $(S*G)$-mod $\simeq$ A-mod.

<u>Claim</u> $A = \mathrm{End}_R(\bigoplus_i M_i)$ is a "path algebra" over k



E.g. $G = \mathbb{C}^2/\mathbb{Z}_2$

$\mathbb{Z}_2$ is gen by g $\quad g(s,t) = (-s,-t)$.

$$R = k[s,t]^G = \frac{k[x,y,z]}{(xy - z^2)}. \qquad \begin{array}{l} x = s^2 \\ y = t^2 \\ z = st. \end{array}$$

$$S = k[s,t] = \bigoplus_i M_i \otimes \rho$$
$$= (\underbrace{M_0 \otimes 1}_{\text{even}}) \oplus (\underbrace{M_1 \otimes \rho_1}_{\text{odd}}) \text{ degree.}$$

M_0 and M_1 are R-mod $\qquad M_0$ is R

2-2

M_1 contains s and t. $\qquad R = \dfrac{k[x,y,z]}{(xy - z^2)}$ $\qquad x = s^2,\ y = t^2,\ z = st$

$$\cdots \xrightarrow{\begin{pmatrix} y & -z \\ -z & x \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} x & z \\ z & y \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} y & -z \\ -z & x \end{pmatrix}} R^{\oplus 2} \xrightarrow{(s,t)} M_1 \to 0.$$

$$\begin{pmatrix} x & z \\ z & y \end{pmatrix}\begin{pmatrix} y & -z \\ -z & x \end{pmatrix} = (xy - z^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$M_1 = \operatorname{coker}\begin{pmatrix} y & -z \\ -z & x \end{pmatrix}.$$

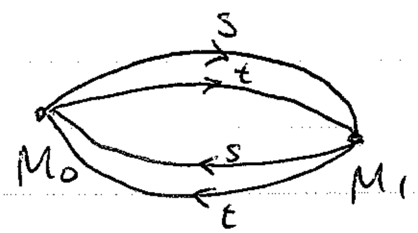
$$M_0 = R.$$

A is $\operatorname{End}_R(M_0 \oplus M_1)$.

$\operatorname{Hom}_R(M_0, M_0) = R = \operatorname{Hom}_R(M_1, M_1)$

$\operatorname{Hom}_R(M_0, M_1) = M_1 \qquad (sR \text{ or } tR).$

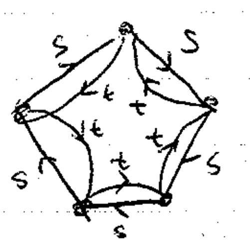
$\operatorname{Hom}_R(M_1, M_0) = M_1$



We impose a relation
$st = ts$.
"quiver with relations"

We have proven that a G-equi sheaf is equi to a rep of the Mckay quiver <u>with relation</u>.

$\mathbb{C}^2/\mathbb{Z}^5$



$st = ts$.

Subtlety :

Let e_i be the zero length path in A for node i.

Consider Ae_i as a left A-mod.

(Space generated by all path starting at node i)

$\mathrm{Hom}_A(Ae_i, Ae_j)$ is gen by paths. $j \to i$.

We may identify $M_i = Ae_i$ <u>if we reverse paths</u>.

$A = \mathrm{End}_R(\oplus M_i)$

A is path alg of quiver Q.

M_i is identified with Ae_i

An arrow $i \to j$ corr to a map in $\mathrm{Hom}(M_j, M_i)$.

Exercise Do for D_4.

D_4 is the 8 elt gp. gen by $g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

acts on $\begin{pmatrix} s \\ t \end{pmatrix}$

$x = st(s^4 - t^4)$ $y = s^4 + t^4$ $z = s^2t^2$.

Crepant Resolution. $\pi : X \longrightarrow Y$.

$\pi^* K_Y = K_X$.

~~If we have a top nonvanishing~~

We want X to have trivial canonical class.

Mckay correspondence

G-eq sheaves on $\mathbb{C}^n/G$ sheaves on Crepant blowup

E.g. $\mathbb{C}^2/\mathbb{Z}_2$ blow up is $\mathcal{O}_{\mathbb{P}^1}(-2)$.

Consider coherent sheaves on $\mathbb{P}^1$ w/ homog. coord $[x_0, x_1]$

Let M be a graded $k[x_0, x_1]$-module.

If $M = k$ ann by x_0, x_1.

Then the coherent sheaf $\tilde{M} = 0$

Any sheaf that "would be" supported only at $x_0 = x_1 = 0$ is trivial.

(HW) <u>Thm</u> The modules corr. to sheaves supported only at $x_0 = x_1 = 0$ are precisely those annihilated by a suff high power of $\mathfrak{m} = (x_0, x_1)$.

Let $S = k[x_0, x_1]$ hom. coord. ring.

Let $\mathcal{A}$ be subcat of S-mod ann. by $\mathfrak{m}^N$.

$\mathcal{A}$ is "dense", i.e

If $0 \to M' \to M \to M'' \to 0$ then

$$M \in \mathcal{A} \Leftrightarrow M', M'' \in \mathcal{A}.$$

Let S be a morphism in S-mod. s.t.
$\ker(S)$ and $\operatorname{coker}(S)$ is in of
Then $\mathrm{Coh}(\mathbb{P}^1)$ is the ctg of graded f.g
S-modules obtained by quotienting by such morphism

S is an S-module
Define $S = \bigoplus_r S_r$ as the graded decomp.
and $S(r)$ such that $S(r)_n = S_{r+n}$.

So $S(1)$ has dim 2 in grade 0
- - - - - 1 - - -1

A morphism in ctg of graded S-mod.
is def'd to be graded 0

E.g

$$S \xrightarrow{\times x_0} S(1)$$

Consider

$$0 \to S \xrightarrow{\binom{-x_0}{x_1}} S(1)^{\oplus 2} \xrightarrow{(x_0, x_1)} S(2) \to k(2) \to 0. \text{ exact}$$

As coh. sheaves on $\mathbb{P}^1$, we get $(S = \mathcal{O})$

$$0 \to S \xrightarrow{\binom{-x_1}{x_0}} S(1)^{\oplus 2} \xrightarrow{(x_0, x_1)} S(2) \to 0.$$

So $S(2)$ can be written as a resolution in S, $S(1)$

Sim. $\quad 0 \to S(1) \to S(2)^{\oplus 2} \to S(3) \to 0$.

So $S(3)$ can be written as res. in $S, S(1)$

Sim. true for $S(r)$ $\quad r \geq 2$.

Every sheaf has a finite resolution.

$$\cdots \to \bigoplus_r S(r)^{\oplus b_{r,1}} \to \bigoplus_r S(r)^{\oplus b_{r,0}} \to M \to 0$$

by Hilbert Syzygy thm.

$$0 \to S(-1) \to S^{\oplus 2} \to S(1) \to 0$$

Sim. $S(<0)$ can written injectively in terms of S and $S(1)$

We <u>almost</u> have $S \oplus S(1)$ is a generator for $\widehat{\mathrm{coh}(\mathbb{P}^1)}$

$$A = \mathrm{End}(S \oplus S(1))$$

$$\begin{tikzcd} S \arrow[r, <-, bend left, "x_0"] \arrow[r, <-, bend right, "x_1"'] & S(1) \end{tikzcd}$$

We solve this by going to the <u>derived</u> ctg.

<u>Thm</u> $D^b(\mathrm{coh}(X))$ is equi to $D^b(A\text{-mod})$.

(Beilingsen)

Derived Morita Equi

An equi bet'n $D^b(S\text{-mod})$ and $D^b(R\text{-mod})$

Let T be an object in $D^b(S\text{-mod})$ that is a "tilting object"

(copies of summands of T must generate all of $D^b(S\text{-mod})$ by iterated mapping cones

$$A = \mathrm{Hom}_{D(S)}(T, T)$$

we must have $\mathrm{Ext}^n(T, T) = 0$ for all $n \neq 0$.
$= \mathrm{Hom}_{D(S)}(T, T[n])$)

For $\mathbb{P}^1$,

$T = S \oplus S(1)$

$\mathrm{Ext}^n(S(a), S(b)) = \mathrm{Ext}^n(S, S(b-a)) = H^n(\mathbb{P}^1, S(b-a))$.

$\Rightarrow \mathrm{Ext}^1(S, S(1)) = \mathrm{Ext}^1(S(1), S) = 0$.

What about $X = \mathcal{O}_{\mathbb{P}^1}(-2)$?

Let $[p, x_0, x_1]$ be hom coord.

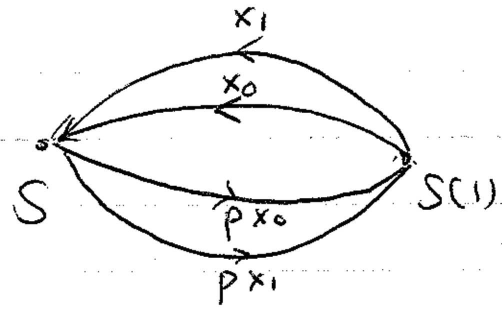
remove $[p, 0, 0]$

and identify $[p, x_0, x_1] \sim [\lambda^{-2}p, \lambda x_0, \lambda x_1]$

deg $-2 \quad 1 \quad 1$

$$0 \to S \to S(1)^{\oplus 2} \to S(2) \to \underbrace{k[p](2)}_{0 \text{ as a sheaf}} \to 0.$$

One can show $\mathrm{Ext}^1(S, S(1)) = \mathrm{Ext}^1(S(1), S)) = 0$.



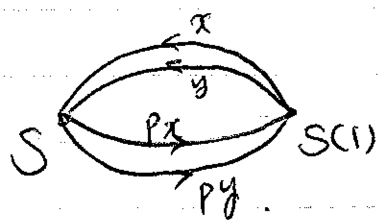
w/. obvious relation.

same quiver (proved McKay corr. for $\mathbb{C}^2/\mathbb{Z}_2$)

<u>3-1</u> $\mathbb{C}^2/\mathbb{Z}_2$. $\quad S = k[\overset{-2}{p}, \overset{1}{x}, \overset{1}{y}]$ $\quad T = \oplus M_i$ $(T = A)$

Tilting Object is $T = S \oplus S(1)$ $\quad \mathcal{M}_i = Ae_i$

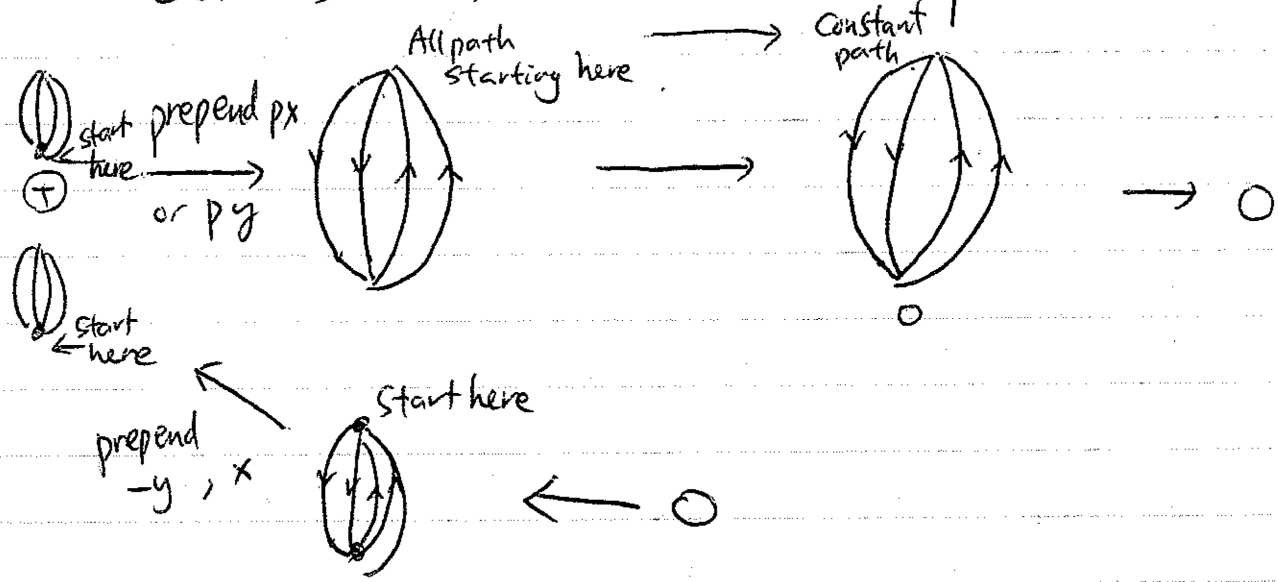
all path starting at node i.



M_i are projective A-mod

S is $\mathcal{O}$ - str. sheaf of $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}$

$S(1)$ is $\mathcal{O}(1)$



$$0 \to S \xrightarrow{\binom{-y}{x}} S(1)^{\oplus 2} \xrightarrow{(px,\, py)} S \longrightarrow V_{(1,0)} \longrightarrow 0.$$

$V_{(1,0)}$ is the complex

$$S \xrightarrow{\binom{-y}{x}} S(1)^{\oplus 2} \xrightarrow{(px,py)} \underline{S}$$

$\leftarrow$ position 0

geometrically we have.

$$\begin{array}{ccccc} \mathcal{O} & \xrightarrow{\binom{-y}{x}} & \mathcal{O}(1)^{\oplus 2} & \xrightarrow{(px,\,py)} & \overline{\mathcal{O}} \\ \uparrow \text{id} & & \uparrow \text{id} & & \uparrow p \\ \mathcal{O} & \xrightarrow{\binom{-y}{x}} & \mathcal{O}(1)^{\oplus 2} & \xrightarrow{(x,y)} & \mathcal{O}(2) \end{array}$$

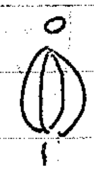
is exact ($\mathcal{O}$ in $D(X)$)

In $D(X)$ $\mathcal{O}(2)$ is equi. to $\mathcal{O} \to \mathcal{O}(1)^{\oplus 2}$

So we have $\mathcal{O}(2) \xrightarrow{p} \underline{\mathcal{O}}$ 1

But $0 \to \mathcal{O}(2) \xrightarrow{p} \mathcal{O} \to \mathcal{O}_E \to 0$ is exact.

E is exceptional $\mathbb{P}^1$

Similarly for 

$$0 \to S(1) \xrightarrow{\binom{-py}{px}} S^{\oplus 2} \xrightarrow{(x,y)} S(1) \to V_{(0,1)} \to 0.$$

geometrically

$$\begin{array}{ccccc} \mathcal{O}(1) & \xrightarrow{\binom{-py}{px}} & \mathcal{O}^{\oplus 2} & \xrightarrow{(x,y)} & \mathcal{O}(1). \\ p \downarrow & & \text{id} \downarrow & & \downarrow \text{id.} \\ \mathcal{O}(-1) & \xrightarrow{\binom{-y}{x}} & \mathcal{O}^{\oplus 2} & \xrightarrow{(x,y)} & \mathcal{O}(1) \end{array}$$

: exact.

In $D(X)$ $\underline{\mathcal{O}(-1)}$ is equi to $\underline{\mathcal{O}}^{\oplus 2} \to \mathcal{O}(1)$

$$\mathcal{O}(1) \to \mathcal{O}(-1) \to \underline{\mathcal{O}}$$

$$0 \to \mathcal{O}(1) \to \mathcal{O}(-1) \to \mathcal{O}_E(-1) \to 0 \quad \text{exact.}$$

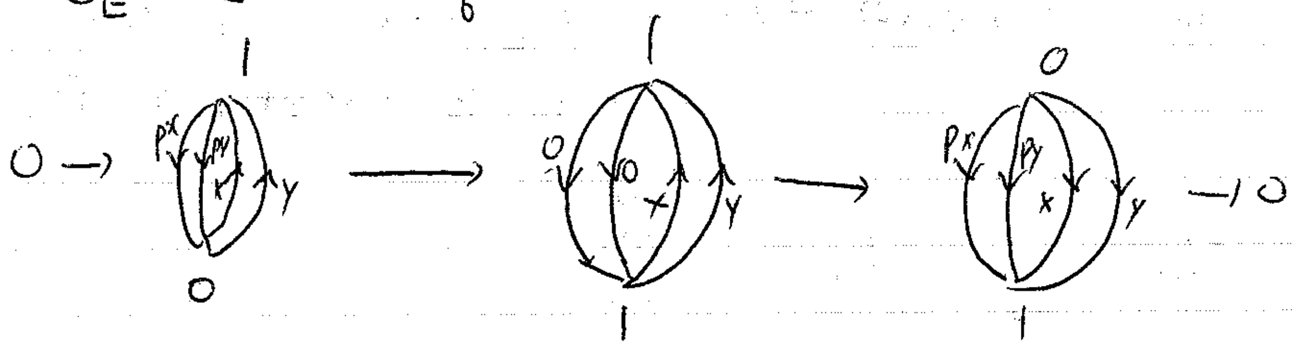
So. $V_{(0,1)}$ is $\mathcal{O}_E(-1) \to \underline{\mathcal{O}}$ a.k.a. $\boxed{\mathcal{O}_E(-1)[1]}$

<u>E.g</u>

Skyscraper sheaf at a point. $q \in E$.

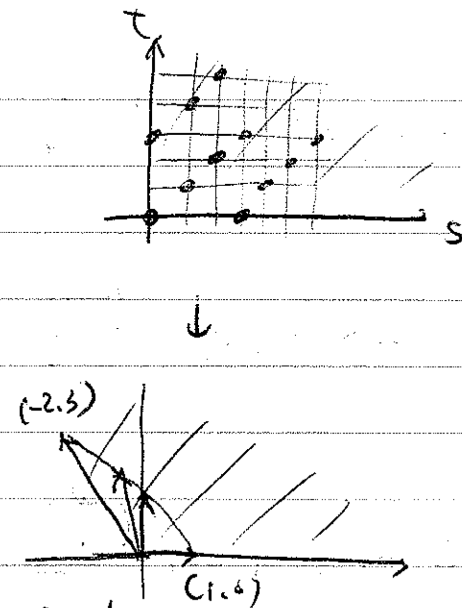
$$0 \to \mathcal{O}_E(-1) \to \mathcal{O}_E \to \mathcal{O}_q \to 0$$

$$\begin{tikzcd} & \mathcal{O}_E \arrow[dr] & \\ \mathcal{O}_E(-1)[1] \arrow[ur, "{[1]}"] & & \mathcal{O}_q \arrow[ll] \end{tikzcd}$$



$\mathbb{C}^2/\mathbb{Z}_3$ $\quad g\cdot(s,t) = (\omega s, s^2 t)$

$\mathbb{C}^2/\mathbb{Z}_3 = \mathrm{Spec}\,(k[st, s^3, t^3])$



Cox hom. coord ring:

Pts at end of rays $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \\ -2 & 3 \end{pmatrix}$

Kernel of transpose is $\begin{array}{cccc} a & b & c & d \end{array}$ $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}$

Interpret this as multidegrees of hom. coord.

i.e. we have hom coords (a, b, c, d)

with 2 $\mathbb{C}^*$ actions $(\lambda a, \lambda^{-2} b, \lambda c, d)$

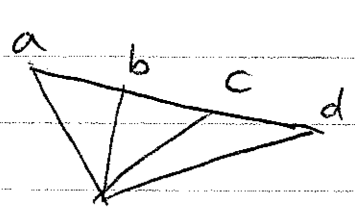
$(a, \mu b, \mu^{-2} c, \mu d)$

3-2. Let Σ be a fan with cones σ

Let the hom coords be $x_1, x_2, \cdots$

Define $S = k[x_1, x_2, \cdots]$

$$B_\Sigma = \Big(\prod_{n\notin\sigma_1} x_n,\ \prod_{n\notin\sigma_2} x_n, \cdots\Big)$$



$$B_\Sigma = (cd, ad, ab)$$
$$= (a,c)\cap(a,d)\cap(b,d)$$

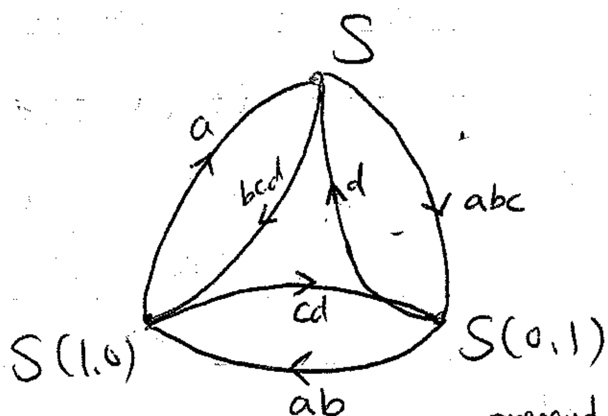
$I_\Sigma = (ac, ad, bd)$ (Stanley-Reisner)

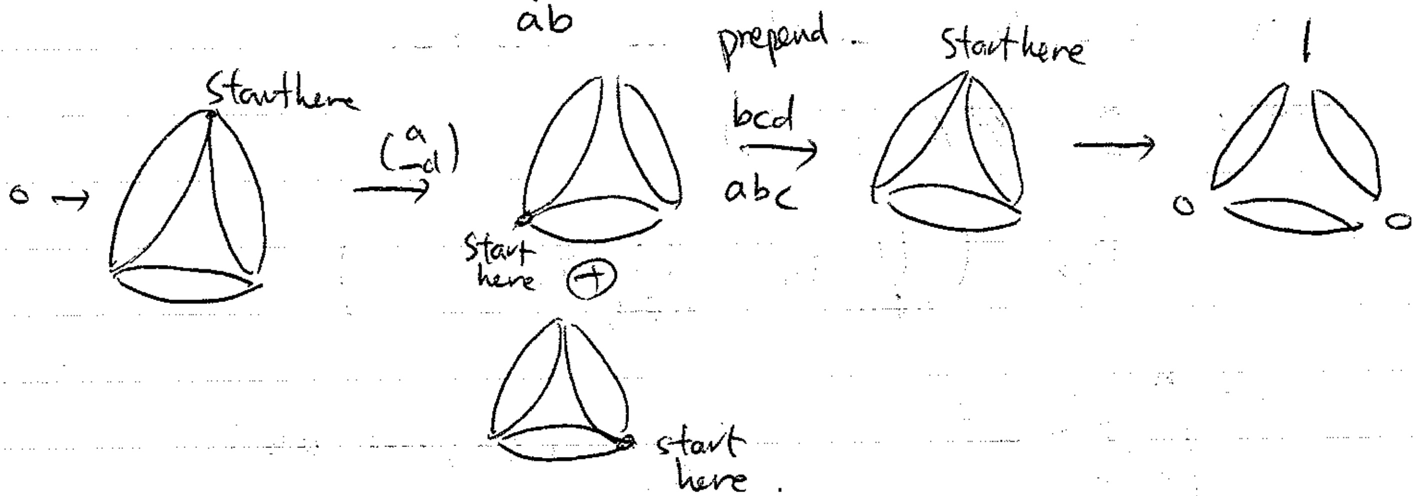
$$0 \longrightarrow S(-2,0) \xrightarrow{\begin{pmatrix} -c \\ a \end{pmatrix}} \begin{matrix} S(-1,0) \\ \oplus \\ S(-1,2) \end{matrix} \xrightarrow{(a,c)} S \longrightarrow 0$$

$$0 \to S(0,-2) \xrightarrow{\begin{pmatrix} -b \\ d \end{pmatrix}} \begin{matrix} S(0,-1) \\ \oplus \\ S(1,-2) \end{matrix} \xrightarrow{(d,b)} S \to 0$$

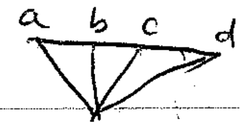
$$0 \to S(-1,-1) \xrightarrow{\begin{pmatrix} -d \\ a \end{pmatrix}} \begin{matrix} S(0,-1) \\ \oplus \\ S(-1,0) \end{matrix} \xrightarrow{(a,d)} S \to 0.$$

$\boxed{S,\ S(1,0),\ S(0,1)}$ will do.



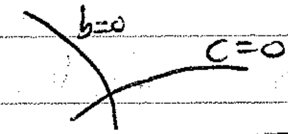


$$S \xrightarrow{\begin{pmatrix} a \\ -d \end{pmatrix}} \begin{matrix} S(1,0) \\ \oplus \\ S(0,1) \end{matrix} \xrightarrow{(bcd,\, abc)} S \longrightarrow V_{(1,0,0)}.$$

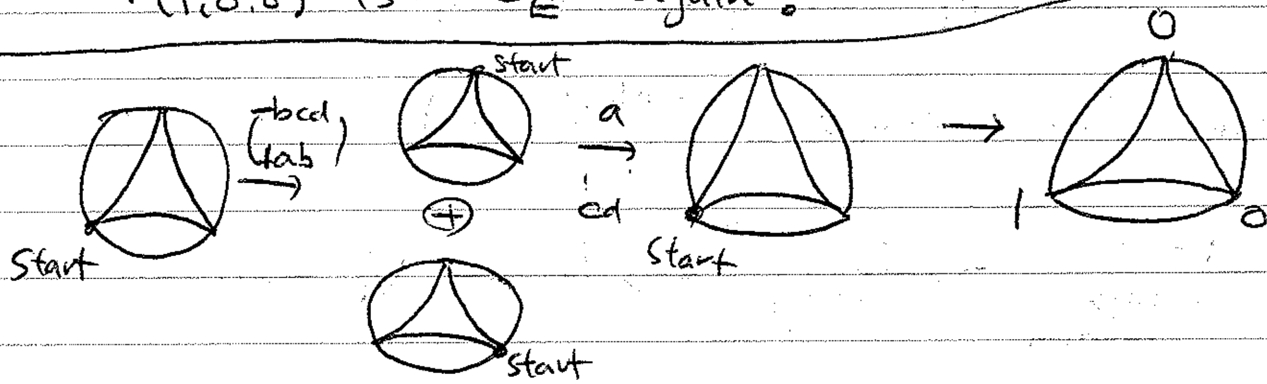


Using the last exact seq. we get

$$S(1,1) \xrightarrow{bc} S \longrightarrow V_{(1,0,0)}$$



$V_{(1,0,0)}$ is $\mathcal{O}_E$ again!



$$S(1,0) \xrightarrow{\binom{-bcd}{+ab}} \begin{matrix} S \\ \oplus \\ S(0,1) \end{matrix} \xrightarrow{(a,cd)} S(1,0) \longrightarrow V_{(0,1,0)}$$

$(a, cd) = (a,c) \cap (a,d) \supseteq B \Rightarrow$ cokernel is 0.

In other words,

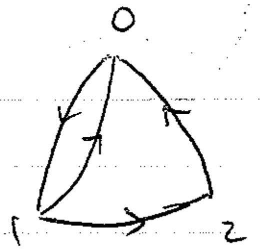
$$S(-2,1) \xrightarrow{\binom{-cd}{a}} \begin{matrix} S(-1,0) \\ \oplus \\ S(-1,1) \end{matrix} \xrightarrow{(a,cd)} S$$ is exact.

$$V_{(0,1,0)} = S(1,0) \xrightarrow{b} S(-1,1) \to \underline{0} = \mathcal{O}_{C_1}(-1,1)[1].$$

$V_{1,0,0} = \mathcal{O}_E$

$V_{0,1,0} = \mathcal{O}_{C_1}(-1,1)[1]$

$V_{0,0,1} = \mathcal{O}_{C_2}(1,-1)[1]$



Suppose V_0, V_1, V_2 are the "simple" (1-dim) quiver rep
P_0, P_1, P_2 are projective reps

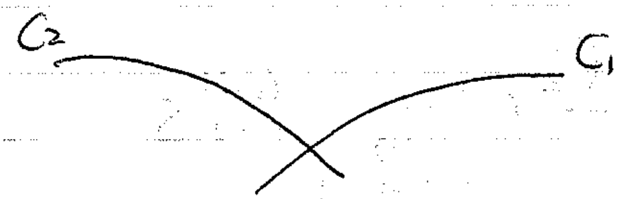
Suppose we compute $\mathrm{Ext}^1(V_i, V_j)$

$$\begin{matrix}\text{stuff to} \\ \text{do with} \\ \text{relation}\end{matrix} \longrightarrow \bigoplus P_j^{\oplus n_{ij}} \longrightarrow P_i \to V_i \to 0$$

$$\mathrm{Hom}(-, V_j) \quad \cdots \leftarrow \mathbb{C}^{\oplus n_{ij}} \longleftarrow 0$$

$$\mathrm{Ext}^1(V_i, V_j) = \mathbb{C}^{\oplus n_{ij}}$$

$\widetilde{\mathbb{C}^2/\mathbb{Z}_3}$



$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{C_1}, \mathcal{O}_{C_2})\ ?$$

$$0 \to \mathcal{O}_X(-C_1) \to \mathcal{O}_X \to \mathcal{O}_{C_1} \to 0$$

$$H^0(X, \mathcal{O}_{C_2}(C_1)) \to \underset{\mathbb{C}^{m+1}}{\mathrm{Hom}}(\mathcal{O}_X(-C_1), \mathcal{O}_{C_2}) \leftarrow \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_{C_2}) \overset{=\mathbb{C}}{\leftarrow} \underbrace{\mathrm{Hom}(\mathcal{O}_{C_1}, \mathcal{O}_{C_2})}_{=0}$$

$$\mathrm{Ext}^1(\mathcal{O}_X(-C_1), \mathcal{O}_{C_2}) \leftarrow \underbrace{\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_{C_2})}_{= H^1(\mathcal{O}_{C_2}) = 0} \leftarrow \underbrace{\mathrm{Ext}^1(\mathcal{O}_{C_1}, \mathcal{O}_{C_2})}_{= \mathbb{C}^m} \leftarrow$$

$\boxed{C_1 \cap C_2 = m \text{ pts.}}$

Del Pezzo Surface & Exceptional collections

- **Def** Del Pezzo surface is sm rational Fano surface V. K_V the canonical class, ($-K_V$ is ample)

Example
- $\mathbb{P}^2$ $\quad K_{\mathbb{P}^2} = \mathcal{O}(-3)$
- $\mathbb{P}^1 \times \mathbb{P}^1$ $\quad K_{\mathbb{P}^1\times\mathbb{P}^1} = \mathcal{O}(-2,-2)$

) minimal surfaces

- B_r – blow up r pts in $\mathbb{P}^2$ in general position. $(r \leq 8)$

$$B_r = Bl_{\{p_1 \cdots p_r\}} \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$$

$$K_{B_r} = \underbrace{-3H}_{=\pi^* K_{\mathbb{P}^2}} + \sum_{i=1}^r E_i$$

$$K_{B_r}^2 > 0 \Rightarrow K_{B_r} = (3H)^2 + \sum_{i=1}^r E_i^2 = 9 - r$$

- Classification.

Key invariant the degree of V : $d = K_V^2$.

<u>property</u> V is Del Pezzo then $\operatorname{rk} \operatorname{Pic}(V) + d = 10$

$$V' = Bl_p V \xrightarrow{\pi} p \in V$$

- $\operatorname{Pic} V' = \operatorname{Pic} V \oplus E$.
- $K_{V'} = \pi^* K_V + E$.

$\Rightarrow K_{V'}^2 = K_V^2 + E^2 = K_V^2 - 1$.

$\Rightarrow \operatorname{rk}\operatorname{Pic}(V') + K_{V'}^2 = \operatorname{rk}\operatorname{Pic}(V) + K_V^2$.

Check	$\mathbb{P}^2$	$\mathbb{P}^1\times\mathbb{P}^1$ ← minimal
rk Pic	1	2
K^2	9	$(-2)^2+(-2)^2 = 8$.

$$\Rightarrow \boxed{1 \leq d \leq 9.}$$

Thm (classification) V Del Pezzo. then

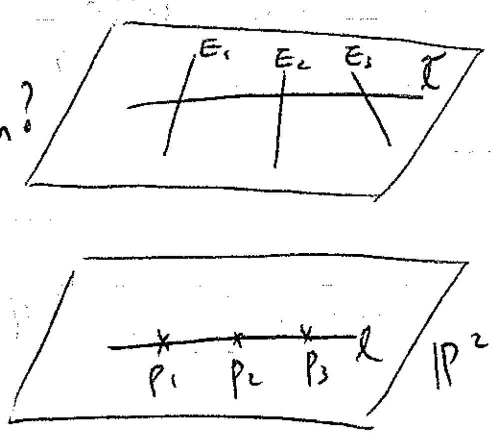
i) $V = \mathbb{P}^1 \times \mathbb{P}^1$ or

ii) $V = B_r$ for $0 \le r \le 8$.

The converse is also true for ii) for $r \le 6$ pts in general position.

(no 3 pts lie on a line
no 6 pts lie on a conic.

why general position?



$$\tilde{\ell} = \pi^*\ell - E_1 - E_2 - E_3$$

$$\tilde{\ell}^2 = \ell^2 + E_1^2 + E_2^2 + E_3^2$$

$$= 1 - 3 = -2.$$

V Del Pezzo. $C \hookrightarrow V$ is irreducible $C^2 < 0$.

$$\underset{\underset{-2}{\vee\!\mid}}{2p_a(C) - 2} = (K_V + C)\cdot C = -\underset{\ulcorner 0}{(-K_V)\cdot C} + \underset{\ulcorner 0}{C^2} < 0$$

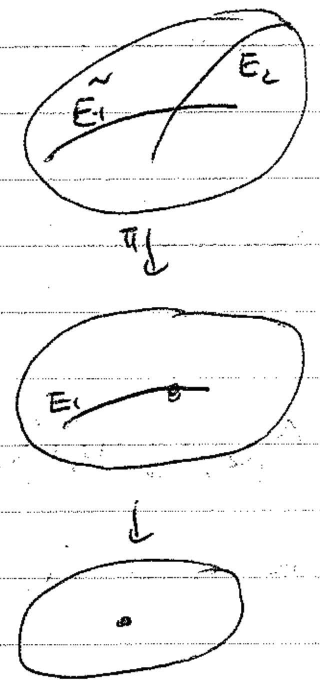
$\Rightarrow p_a(C) = 0$, $C^2 = -1$ $\Rightarrow$ C is (-1)-rat'l curve. //

V Del Pezzo.

$\downarrow$ by succesive blow-up's

$\mathbb{P}^2$.

If some pt on the exceptional curve is blown up.



$$\tilde{E}_1 = \pi^* E_1 - E_2$$

$$\Rightarrow \tilde{E}_1^2 = E_1^2 + E_2^2 = -2.$$

This cannot happen

- Exceptional collections.

Goal Describe $D^b(V)$

Def $E \in D^b(V)$ is called <u>exceptional</u> if

$\mathrm{Hom}(E, E) = k$. $\mathrm{Hom}^i(E, E) = 0$ if $i \neq 0$.

Def $\langle E_1, E_2, \cdots, E_n \rangle$ is called Exceptional collection

if 1) E_i is exceptional

2) $\mathrm{Hom}^\bullet(E_i, E_j) = 0$ if $i > j$

$(E_1 \cdots E_n)$ is called complete "exceptional collection"

if $\langle E_1 \cdots E_n \rangle = D^b(V)$

e.g. $\mathbb{P}^1$ $\langle \mathcal{O}, \mathcal{O}(1) \rangle$ complete.

$\mathbb{P}^2$ $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$

Thm (Orlov) Let $\begin{array}{c} V' = Bl_p V \\ \pi \downarrow \\ V \end{array}$ and $(E_1 \cdots E_n)$ complete exceptional collection on V.

Then $(\mathcal{O}_E(-1), E_1 \cdots E_n)$ is complete exceptional coll. on V'.

Cor $\begin{array}{c} B_r = V \\ \downarrow \\ \mathbb{P}^2 \end{array}$ then $(\mathcal{O}_{E_1}(-1), \cdots, \mathcal{O}_{E_r}(-1), \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2))$

E_i : exceptional curve over p_i.

$\forall F_i \in D^b(V)$

$$\begin{array}{ccc} E & \xhookrightarrow{j} & V' \\ p\downarrow & & \downarrow \pi \\ p & \xhookrightarrow{i} & V \end{array}$$

$$\begin{aligned} &\mathrm{Hom}^\bullet(\pi^* F_i, \mathcal{O}_E(-1)) \\ &= \mathrm{Hom}^\bullet_{V'}(\pi^* F_i, j_* \mathcal{O}_E(-1)) \\ &= \mathrm{Hom}^\bullet_E(j^* \pi^* F_i, \mathcal{O}_E(-1)) \\ &= \bigoplus \mathrm{Hom}^\bullet_E(\mathcal{O}_E, \mathcal{O}_E(-1)) = 0. \end{aligned}$$

- $\mathcal{O}_E(-1)$ is exceptional.

$$\begin{aligned} &\mathrm{Hom}^i_{V'}(j_* \mathcal{O}_E(-1), j_* \mathcal{O}_E(-1)) \\ &= \mathrm{Hom}^i_E(j^* j_* \mathcal{O}_E(-1), \mathcal{O}_E(-1)) \end{aligned}$$

$j^* \mathcal{O}_{V'}(E) = \mathcal{O}_{V'}(E)|_E = \mathcal{O}_E(-1)$ $\quad E : (-1)$-curve.

$\Rightarrow j_* \mathcal{O}_E(-1) = \mathcal{O}_{V'}(E) \otimes j_* \mathcal{O}_E = \mathcal{O}_{V'}(E)|_E$

$$0 \to \mathcal{O}_{V'} \to \mathcal{O}_{V'}(E) \to \mathcal{O}_{V'}(E)|_E \to 0.$$

$$\Rightarrow j^* j_* \mathcal{O}_E(-1) = j^* ([\mathcal{O}_{V'} \to \mathcal{O}_{V'}(E)])$$

$$= [\mathcal{O}_E \longrightarrow \mathcal{O}_E(-1)] = \mathcal{O}_E(-1) \oplus \mathcal{O}_E[1]$$

$$\Rightarrow \mathrm{Hom}^i_E(j^* j_* \mathcal{O}_E(-1), \mathcal{O}_E(-1)) = \mathrm{Hom}^i(\mathcal{O}_E(-1) \oplus \mathcal{O}_E[1], \mathcal{O}_E(-1))$$

$$= \begin{cases} k & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

<u>Morita Thm for derived cats and tilting</u>

When do rings R, S have $D(R) \cong D(S)$

<u>Morita Thm</u> (Rickard) TFAE.

1) $D(R) \cong D(S)$

2) There is a "tilting" object $T \in D(S)$ s.t. $\mathrm{Hom}_{D(S)}(T, T) \cong R$.

(flatness) (3) There is a bimodule M s.t. $\overset{L}{\otimes}_R M : D(R) \to D(S)$ is an equi.

Tilting means $F : D(R) \to D(S)$

$$R \in D(R) \longmapsto F(R).$$

$\mathrm{Hom}^\circ(R, R) = R$ $\qquad$ $\mathrm{Hom}^\circ_{D(R)}(F(R), F(R)) = R$

· higher exts $\mathrm{Ext}^1(T, T) = 0$

- T a compact generator of $D(S)$

$\mathrm{Hom}(T, \cdot)$ commutes w/ direct sum.

$T \in D(S)$ is a generator if $\forall\, y \in D(S)$

$\mathrm{Hom}_{D(S)}(T, y[i]) = 0$ for all $i \Rightarrow y = 0$.

this notion is equi to the notion of T generating $D(S)$ via shifts, direct summands, cones, direct sums.

$D(X) = D(\mathrm{Qcoh}\, X) \rightsquigarrow$ compact = bounded cpx. of v.b.

- $\exists$ compact gen. $\overset{\shortparallel}{T}$ via induction on affine patch.

$\dot{A} = \mathrm{End}^{\cdot}_{D(X)}(T) = $ dg Algebra

$D(\text{dg } A\text{-mod}) \cong D(X)$ (Keller)

4-2 S : Del Pezzo surface.

$M_0, M_1, M_2, \cdots$ is an exceptional collection + complete

+ strong $\Leftrightarrow$ $\mathrm{Ext}^i(M_j, M_k) = 0 \quad \forall i \neq 0 \quad \forall j, k$.

$\Rightarrow T = M_0 \oplus M_1 \oplus \cdots$ is a tilting object.

Mutations

$\underline{\mathrm{Hom}(M_0, M_1) \otimes M_0} \longrightarrow M_1 \longrightarrow \text{cone}$

$\langle \text{cone}, M_0, M_2, \cdots \rangle$ is also exceptional.

$D^b(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n) \rangle$

want.

$\mathbb{P}^n = \mathbb{P}V$

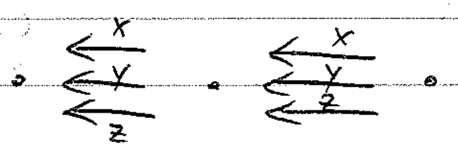
$= \langle \Omega^1(1), \mathcal{O}, \mathcal{O}(2), \cdots \rangle$

$\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) \otimes \mathcal{O} \longrightarrow \mathcal{O}(1)$

$\quad \| \; V^*$

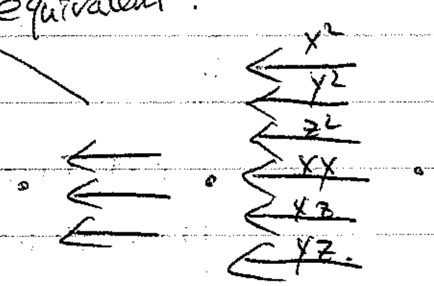
$0 \to \underline{\underline{\Omega^1(1)}} \to \mathcal{O} \otimes V^* \longrightarrow \mathcal{O}(1) \longrightarrow 0$.

$\mathbb{P}^2 \quad \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$



equivalent.

$\langle \Omega^1(1), \mathcal{O}, \mathcal{O}(2) \rangle$



Let X be the normal bundle of Del Pezzo in a CY

i.e. X is the canonical bundle of S

$i : S \hookrightarrow X \qquad \pi : X \to S$

$F_0, F_1 \cdots$ is a strong complete excep coll on S

Try "$T = \pi^* F_0 \oplus \pi^* F_1 \oplus \cdots$"?

Let L_i be the one-dim simple quiver repr (dual to P_i)

The quiver for tilting obj. $\oplus P_i$ is the ext quiver for L_i

Try $i_* L_i$ to form the simple object in X.

Compute $\mathrm{Ext}^1_X(i_* L_i, i_* L_j)$

<u>Problem</u> $\quad i : S \hookrightarrow X \quad \mathcal{E}, \mathcal{F}_i$ sheaves on S

$$\mathrm{Ext}^n_X(i_*\mathcal{E}, i_*\mathcal{F}_i) \qquad \mathrm{Hom}_{D(S)}(i^* i_* \mathcal{E}, \mathcal{F}_i)$$

$$\Uparrow \qquad\qquad R^q i^* i_*$$

$$E_2^{p,q} = \mathrm{Ext}^p_S(\mathcal{E}, (\Lambda^q N) \otimes \mathcal{F}_i)$$

conormal bundle.

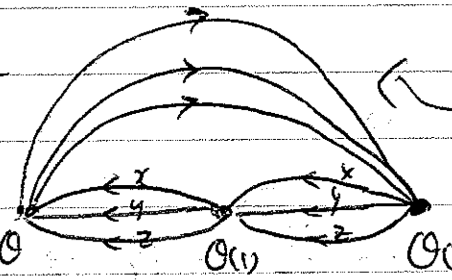
$$\begin{array}{ccc}
\mathrm{Ext}^2(\mathcal{F}_i, \mathcal{E}) & \mathrm{Ext}^1(\mathcal{F}_i, \mathcal{E}) & \mathrm{Hom}(\mathcal{F}_i, \mathcal{E}) \\
\parallel & \parallel & \parallel \\
\mathrm{Hom}_S(\mathcal{E}, \mathcal{F}_i \otimes K) & \mathrm{Ext}^1_S(\mathcal{E}, \mathcal{F}_i \otimes K) & \mathrm{Ext}^2(\mathcal{E}, \mathcal{F}_i \otimes K) \\
\mathrm{Hom}_S(\mathcal{E}, \mathcal{F}_i) & \mathrm{Ext}^1_S(\mathcal{E}, \mathcal{F}_i) & \mathrm{Ext}^2_S(\mathcal{E}, \mathcal{F}_i)
\end{array}$$

At most one of those 6 is nonzero.

$\mathrm{Ext}^1(i_*\mathcal{E}, i_*\mathcal{F}_i) = \mathrm{Ext}^1_S(\mathcal{E},\mathcal{F}_i) \oplus \mathrm{Ext}^2_S(\mathcal{F}_i,\mathcal{E})$ (at most one of the rows is nonzero)

$\uparrow$ relations in quiver

E.g. $\mathbb{P}^2$



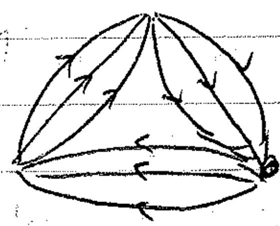
3 relations

Excep. coll in $\mathbb{P}^2$ is $\mathcal{O}$ $\mathcal{O}(1)$ $\mathcal{O}(2)$

$\mathbb{C}^3/\mathbb{Z}_3$ $\quad g : (x,y,z) \longmapsto (\omega x, \omega y, \omega z)$

resolved with $\mathcal{O}_{\mathbb{P}^2}(-3)$

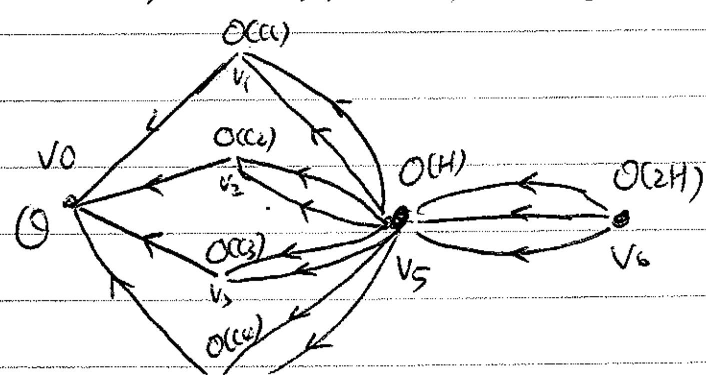
Makay Quiver — 3 irrep. $1, \rho_1, \rho_2$



E.g. dP_4 (5 pts blown up on $\mathbb{P}^2$) — degree 5

A strong exceptional collection is

$(\mathcal{O}, \mathcal{O}(C_1), \mathcal{O}(C_2), \mathcal{O}(C_3), \mathcal{O}(C_4), \mathcal{O}(H), \mathcal{O}(2H))$



$\mathrm{Hom}(L_i, L_j) = k\delta_{ij}$

$\mathrm{Hom}_X(i_*\mathcal{E}, i_*\mathcal{F}_i) = \mathrm{Hom}_S(\mathcal{E}, \mathcal{F}_i) \oplus \mathrm{Ext}^3_S(\mathcal{F}_i, \mathcal{E})$

Proj. resolution of L_6 is

$$0 \to P_0 \to \begin{matrix} P_1 \\ \oplus \\ P_2 \\ \oplus \\ P_3 \\ \oplus \\ P_4 \end{matrix} \to P_5^{\oplus 3} \to P_6 \to L_6 \to 0$$

$\Rightarrow \mathrm{Ext}^3(L_6, L_0) = k$. $\Rightarrow$ need to add arrows.

If S can be contracted to point

Let Q be the quiver asso. to $D(K_S)$

then # arrow going into a vertex

= # arrow going out. (Anomalies)

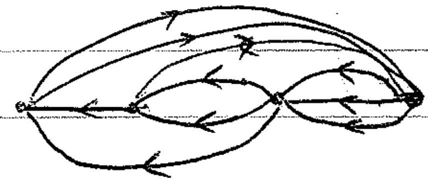
if $m_i = 1$ $\forall$ nodes

Define a quiver with multiplicities : m_i

McKay : Dim of ireps $= m_i$

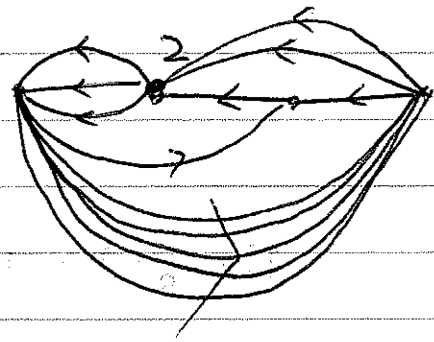
If $T = \oplus$ (Vector bundles) m_i = ranks of bundles

Eg. dP_1 $\quad \mathcal{O}, \mathcal{O}(C), \mathcal{O}(H), \mathcal{O}(2H)$



arrows $\rightarrow$ are added in canonical bundle of dP_1. (CY3)

Mutation



Mckay Corr for $\mathbb{C}^2/D_4$ $\quad D_4$ gen by $g=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $h=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ act on s and t.

$$x = st(s^4-t^4) \qquad y = s^4+t^4 \qquad z = s^2t^2$$

$$f = x^2 - z(y^2-4z^2) = 0.$$

$$k[s,t] = \bigoplus M_i \otimes \rho_i \qquad R = k[s,t]^G = \frac{k[x,y,z]}{f}$$

$M_0 = R$

M_1 is gen by $\alpha = st(s^2-t^2)$ $\quad \beta = -(s^2+t^2)$.

$$\rho_1(g) = -1 \qquad \rho_1(h) = 1$$

$$M_1 = \operatorname{coker}\begin{pmatrix} x & z(y-2z) \\ y+2z & x \end{pmatrix}$$

$$M_2 = \operatorname{coker}\begin{pmatrix} x & z(y+2z) \\ y-2z & x \end{pmatrix}$$

$$M_3 = \operatorname{coker}\begin{pmatrix} x & z \\ y^2-4z^2 & x \end{pmatrix}.$$

$$M_4 \text{ is } \operatorname{coker}\begin{pmatrix} x & y & -z & 0 \\ yz & x & 0 & -z \\ 4z^2 & 0 & x & -y \\ 0 & 4z^2 & -yz & x \end{pmatrix}$$

Corr. to

$$\begin{pmatrix} s \\ -s^2t \\ -2t^3 \\ 2st^2 \end{pmatrix} \quad \begin{pmatrix} t \\ st^2 \\ 2s^3 \\ 2s^4t \end{pmatrix}$$

$T = M_0 \oplus \cdots \oplus M_4$

$\operatorname{Hom}_R(M_i, M_j)$.

$$\begin{tikzcd}
 & M_0 \arrow[d, bend right, "{(-t,s)}"'] & \\
M_3 \arrow[r, bend left, "{(t,s)}"] & M_4 \arrow[l, bend left, "{\binom{-s}{t}}"] \arrow[u, bend right, "{\binom{s}{t}}"'] \arrow[r, bend right, "{\binom{t}{-s}}"'] \arrow[d, bend right, "{\binom{t}{s}}"'] & M_1 \arrow[l, bend right, "{(-s,-t)}"'] \\
 & M_2 \arrow[u, bend right, "{(-s,t)}"'] &
\end{tikzcd}$$

Bridgeland Stability.

$\mathcal{A}$: abelian ctg.

<u>Def</u> A stability ftn : $Z : K(\mathcal{A}) \to \mathbb{C}$ is a gp hom.

s.t. $0 \neq E \in \mathcal{A}$ $\quad Z(E) = m(E)\exp(i\pi\phi)$ $\quad \phi \in (0,1]$

$m(E) \in \mathbb{R}_{>0}$.

ϕ is called the phase.

$0 \neq E \in \mathcal{A}$ then E is semistable if for all $G \subset E$ we have $\phi(G) \leq \phi(E)$

Z has the Harder-Narasimhan filtration if

$\exists$ a filtration of the form

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E.$$

$F_i = E_i/E_{i-1}$ are semistable & $\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n)$

<u>Def</u> If $\mathcal{D}$ is triangulated then a stability condition $\sigma = (Z, \mathcal{P})$ is a collection of full additive subctg $\mathcal{P}(\phi) \subset \mathcal{D}$ $\phi \in \mathbb{R}$. s.t.

(1) $\phi_1 > \phi_2 \Rightarrow \mathrm{Hom}(A_1, A_2) = 0$

(2) $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$

(3) HN property : $0 \neq E \in \mathcal{D} \Rightarrow \exists$ a finite # of real numbers $\phi_1 > \cdots > \phi_n$ s.t.

$$0 \neq E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n = E.$$

$A_i \in \mathcal{P}(\phi_i)$

$A_1 \quad A_{n-1} \quad A_n$

(1) – (3) are called <u>slicing</u>.

(4) $Z : K(D) \to \mathbb{C}$ $\quad Z(E) = m(E)\exp(i\pi\phi)$ $\quad \phi \in (0,1]$
gp hom. $\quad E \in P(\phi)$
$m(E) > 0$

$$K^{num}(D) = K(D)/K(D)^{\perp}$$

$$\chi(E,F) = \sum (-1)^i \dim \mathrm{Ext}^i(E,F)$$

$$K(D)^{\perp} = \{ E \in K(D) : \chi(E,F) = 0 \ \ \forall F \in K(D) \}$$

· $K^{num}(D)$ finite rank for $D = D^b(X)$ where X is a smooth proj. var over $\mathbb{C}$.

<u>Thm</u> Numerical stability conditions $\mathrm{Stab}(D)$

$\Rightarrow$ $\mathrm{Stab}(D)$ is a complex mfd

$\sigma = (Z, P) \longrightarrow \mathrm{Hom}(K^{num}(D), \mathbb{C})$

local homeomorphism.

<u>Conjecture</u> $\mathrm{Stab}(X)$ is contractible X is smooth projective variety over $\mathbb{C}$. (for curves)

<u>t-structure</u>

A t-structure on D is full subctg $\mathcal{F} \subset D$ s.t.
$\mathcal{F}[1] \subset \mathcal{F}$ define $\mathcal{F}^{\perp} := \{ G \in D : \mathrm{Hom}(F,G) = 0 \ \forall F \in \mathcal{F} \}$.
$\forall H \in D$ we have a triangle $F \to H \to G$ where $F \in \mathcal{F}$ $G \in \mathcal{F}^{\perp}$

Standard t-structure.

$\mathcal{F}$ is all $F \in D$ s.t. $H^i(F) = 0$ $\forall i > 0$.

The heart

$\mathcal{A} = \mathcal{F} \cap \mathcal{F}[1]^{\perp}$

$\mathcal{A}$: abelian ctg.

It is a bounded t-structure

$$D = \bigcup_{i,j} \mathcal{F}[i] \cap \mathcal{F}[j]^{\perp}.$$

Fact The heart determines the t-structure.

Fact $\mathcal{A}$ is a heart $\Leftrightarrow$

$A_i \in \mathcal{A}$

(1) $k_1 > k_2$ $k_1, k_2 \in \mathbb{Z} \Rightarrow \mathrm{Hom}(A_1[k_1], A_2[k_2]) = 0$.

(2) $\forall\, 0 \neq E \in D$ $\exists\, k_1 > \cdots > k_n$ s.t.

$$\begin{tikzcd}[column sep=small] 0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\ & A_1 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] & \end{tikzcd}$$

$A_i \in \mathcal{A}[k_n]$

Thm A stability condition on D is equivalent to giving a stability condition on a heart of a bounded t-structure.

Pf $\Rightarrow$ $\sigma = (Z, P)$ $\mathcal{A} = P((0,1])$ (extension closed subctg by $P(\phi)$ $\phi \in (0,1]$.

$\mathcal{A}$ is a heart $(P(>0))$

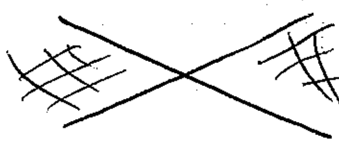
Ex: $D = D^b(E)$ E: Elliptic curve.

$\Rightarrow Z(F) = -\deg(F) + i\,\mathrm{rk}(F)$.

Conifold = ordinary double point.

$xy = zw$ in $\mathbb{C}^4$

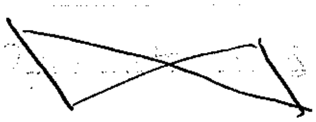
i.e. affine cone on $\mathbb{P}^1 \times \mathbb{P}^1$



$\leftarrow$ not crepant

$$\begin{tikzcd} & Bl_{pt} X \arrow[dl] \arrow[dr] & \\ Y \supset C \arrow[dr] & & C' \subset Y' \arrow[dl] \\ & X & \end{tikzcd}$$

blow up one Weil divisor $= Y \supset C$ $\quad$ $C' \subset Y'$: blowup other

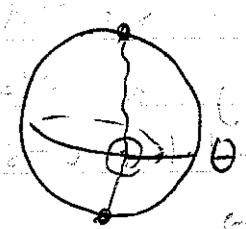


a triangle
$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul, "{[1]}"] & \end{tikzcd}$$
means that A and C can combine to form B

-or- B can decay into A and C.

follow. hep-th/0403166 § 7.2.

Moduli space of stability condition



Bridgeland

$K_{top}(X) \to \mathbb{C}$

$\dim = \sum_n b^{2n}$

$= 2h^{1,1}+2$

physics

$B+iJ \in H^2(X, \mathbb{C})$

$\dim = h^{1,1}$

$N=(2,2)$ SCFT

$\downarrow$ twist

TQFT

B-model or A-model.

b^3 $2h^{1,1}+2$

If $pt \in C$.

$0 \to \mathcal{O}_C(-1) \to \mathcal{O}_C \to \mathcal{O}_{pt} \to 0$.

$$\begin{tikzcd} \mathcal{O}_C \arrow[r] & \mathcal{O}_{pt} \arrow[dl] \\ \mathcal{O}_C(-1)[1] \arrow[u] & \end{tikzcd}$$

$\phi(\mathcal{O}_{pt}) = 0$. $\phi(\mathcal{O}_C(-1)) = -1$

$\phi(\mathcal{O}_C) = \dfrac{\theta}{\pi} - 1$

$0 < \theta < \pi$ $\mathcal{O}_{pt}$ stable

$\theta > \pi$

Where do pts of C' come from?

Compute $\mathrm{Ext}^{\bullet}_X(\mathcal{O}_C, \mathcal{O}_C(-1)) \Leftarrow \mathrm{Ext}^p_C(\mathcal{O}_C, \mathcal{O}_C(-1) \otimes \Lambda^q N)$.

$N_C = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

answer $\mathrm{Ext}^2_X(\mathcal{O}_C, \mathcal{O}_C(-1)) = \mathbb{C}^2$

$\mathbb{P}^1$ of object gotten via

$\mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_C(-1)[2])$

$$\begin{tikzcd} \mathcal{O}_C(-1)[1] \arrow[rr] & & \mathrm{Cone}[-1] \arrow[dl] \\ & \mathcal{O}_C \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Bridgeland math.AG/0009053

$$\begin{tikzcd} Y \arrow[dr, "\pi"'] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

$\pi^* D(X) \to D(Y)$ is an embedding

$R\pi_* L\pi^* \mathcal{O}_X = \mathcal{O}_X$ because rational singularity

$\pi_*(\pi^* F \otimes \mathcal{O}_Y) = F \otimes \pi_*\pi^* \mathcal{O}_X \cong F.$

$D(Y) = \langle \pi^* D(X), \mathcal{C} \rangle \qquad \mathcal{C} := (\pi^* D(X))^{\perp}.$

Toric Picture

$S = k[a, b, c, d]$ $\qquad x = ac \quad y = bd \qquad xy = zw.$

$\qquad\qquad\qquad\qquad\; w = ad \quad z = bc \qquad$ conifold.

grading $\quad 1, 1, -1, -1$

Exclude $a = b = 0$.

$[a, b]$ become homog. coord on $\mathbb{P}^1 = C$.

Exclude $C = d = 0$

$[c, d]$ an homog. coord $\mathbb{P}^1 = C'$

$$0 \to S \xrightarrow{\binom{-b}{a}} S(1)^{\oplus 2} \xrightarrow{(a, b)} S(2) \to 0$$

$T = S \oplus S(1)$.

$$\begin{tikzcd} S \arrow[r, bend left=20, "c"] \arrow[r, bend left=50, "d"'] \arrow[d, dash] & S(1) \arrow[l, bend right=50, "a"'] \arrow[l, bend right=20, "b"] \arrow[d, dash] \\ \mathcal{O}_C & \mathcal{O}_C(-1)[1] \arrow[l, bend left] \end{tikzcd}$$

relation

$acb = bca$

different from $\mathbb{C}^2/\mathbb{Z}_2$

Simple objects are

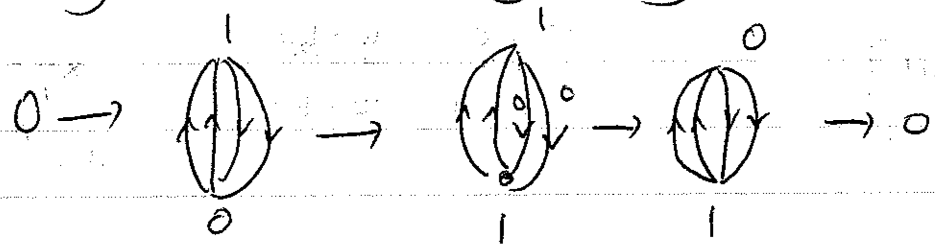
$$D^b(A\text{-mod}) = D^b(X)$$

The abelian ctg A-mod is a possible heart.

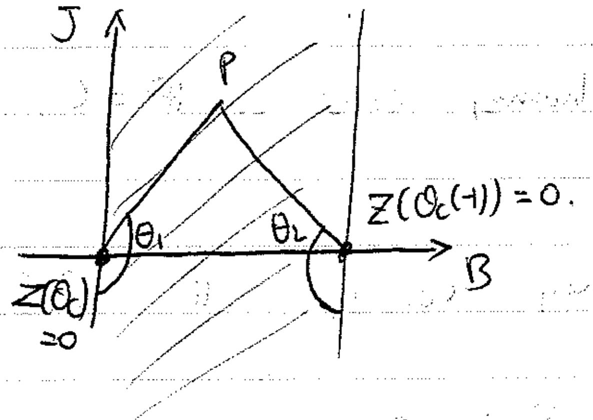
$$0 \to \mathcal{O}_C(-1) \to \mathcal{O}_C \to \mathcal{O}_{pt} \to 0$$

$\Rightarrow$ A pt has dim vector $(1, 1)$

Stability condition is given by a label on each vertex



$Z(F[1]) = -Z(F)$



$Z(a) = m\, e^{i\pi\varphi}$

$\frac{1}{\pi} \arg Z(a) = \varphi(a)$

$\varphi(\mathcal{O}_p) = \frac{1}{2}$ for any p.

$\varphi(\mathcal{O}_c) = \theta_1/\pi$

$\varphi(\mathcal{O}_c(-1)[1]) = \theta_2/\pi$

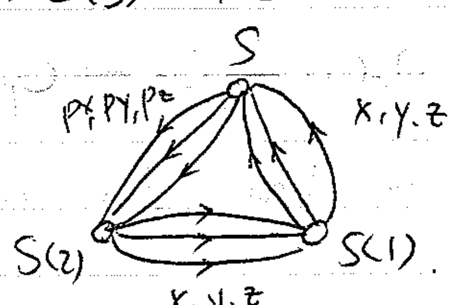
$\mathbb{C}^3/\mathbb{Z}_3$ $\quad$ $S = k[\underset{-3}{p}, \underset{1}{x}, \underset{1}{y}, \underset{1}{z}]$

Remove $x = y = z = 0$

$\widetilde{\mathbb{C}^3/\mathbb{Z}_3}$ is $\mathcal{O}_{\mathbb{P}^2}(-3)$

$$0 \to S \to S(1)^{\oplus 3} \to S(2)^{\oplus 3} \to S(3) \to 0$$

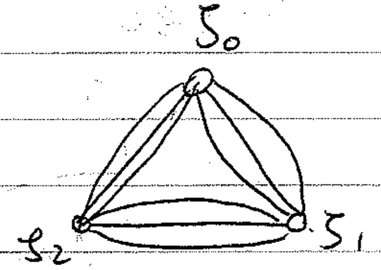
$T = S \oplus S(1) \oplus S(2)$

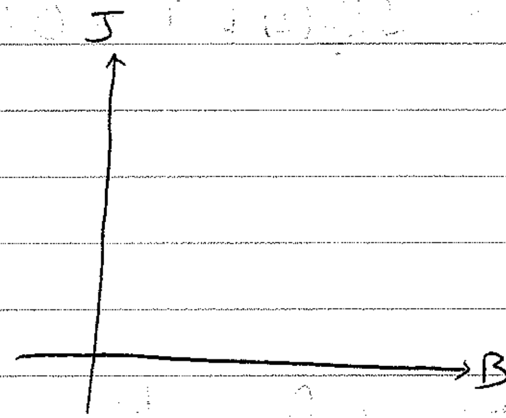


A-module is a heart for Bridgeland stab if

$$\varphi(L_0) = \varphi(L_1) = \varphi(L_2)$$

Ishii(i)
Craw





At LRL

$$Z(F) = \int e^{-(B+iJ)} \, ch(F) \sqrt{td\, T_X} + $$

lowest chern character.

If F is loc. free $\quad Z(F) \sim \frac{1}{6}\int_X (-iJ)^3 = (-i)^3 \, Vol(X)$

$$\varphi(F) = -\frac{3}{2}$$

In general $\quad \varphi(F) = -\frac{1}{2} \dim \operatorname{supp} F$

Coh(X) is <u>not</u> a heart in Bridgeland sense.

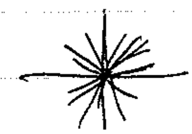
For flop $\qquad\qquad$ At LRL

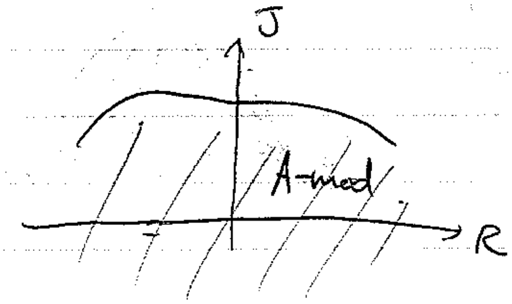
$L_0 = \mathcal{O}_C \qquad \varphi = -\frac{1}{2}$

$L_1 = \mathcal{O}_C(-1)[1] \qquad \varphi = +\frac{1}{2}$

$\mathbb{C}^3/\mathbb{Z}_3$

	At orbifold.	At LRL
$L_0 = \mathcal{O}_E$	$\varphi = 0$	-1
$L_1 = \Omega^1_E(1)[1]$	$\varphi = 0$	$+1$
$L_2 = \Omega^1_E(2)[2] = \mathcal{O}(-1)[2]$	$\varphi = 0$	-1





For a fixed K_{top}-class Q

Let M_Q be the moduli space of stable obj's with class Q

$$\chi_Q = \text{Euler char}(M_Q).$$

$$Z(\vec{s}) = \sum_Q \chi_Q e^{-\vec{Q}\cdot\vec{s}}$$ must vary smoothly with stability condition