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1 Lecture 1

1.1 Keller's Motivation.

Definition: Two complexes $C_1^\bullet, C_2^\bullet$ are said to be *quasi isomorphic* if there exists a third complex $C^\bullet$ and maps

$$\begin{array}{ccccc} & & C^\bullet & & \\ & f \swarrow & & \searrow g & \\ C_1^\bullet & & & & C_2^\bullet \end{array}$$

such that $H^\bullet(f), H^\bullet(g)$ are isomorphisms. We will denote quasi-isomorphism by $C_1^\bullet \overset{q}{\cong} C_2^\bullet$.

Basic Question: When are two complexes quasi isomorphic? This question is similar in nature to the question of when two topological spaces are homotopy equivalent. Two complexes $C_1^\bullet, C_2^\bullet$ are not quasi isomorphic if $H_i(C_1^\bullet) \ncong H_i(C_2^\bullet)$ for some i. But the converse is not true, $H_i(C_1^\bullet) \cong H_i(C_2^\bullet)$ for all i does not imply $C_1^\bullet \overset{q}{\cong} C_2^\bullet$. This is a subtle point, $C_1^\bullet$ is quasi-isomorphic to $C_2^\bullet$ if there *exists* an actual chain map that *induces* an isomorphism on the level of cohomologies.

Counter Example: Let $A = k[x, y]$ and consider the following two complexes of A-modules.

$$\begin{aligned} C_1^\bullet &: \quad 0 \to A^{\oplus 2} \to A \to 0 \\ C_2^\bullet &: \quad 0 \to A \xrightarrow{0} k \to 0 \end{aligned}$$

The map $A^{\oplus 2} \to A$ is defined to be $(f, g) \mapsto xf + yg$. The cohomology groups of these two complexes are isomorphic. We will show that they are not quasi isomorphic. But using the basic definition is not going to work. Instead we use the following characterization.

Claim: Every complex $C^\bullet$ which only has two adjacent non-vanishing cohomology groups, say, in degrees 0 and -1, is determined by $H^0(C^\bullet), H^{-1}(C^\bullet)$ and a class $\alpha \in \mathrm{Ext}^2(H^0(C^\bullet), H^{-1}(C^\bullet))$.

Exercise: Show that in the above claim α is a quasi-isomorphism invariant.

Note that every complex $C^\bullet$ determines a class in $\mathrm{Ext}^2(H^0(C^\bullet), H^{-1}(C^\bullet))$ given by the short exact sequence

$$0 \to H^{-1}(C^\bullet) \to C^{-1}/\mathrm{Im} d^{-2} \xrightarrow{d^{-1}} \ker d^0 \to H^0(C^\bullet) \to 0$$

where $d^i : C^i \to C^{i+1}$ is the differential. By the above claim this class fully characterizes $C^\bullet$ up to quasi-isomorphism.

In this example both complexes have adjacent cohomology groups $H^0(C_i^\bullet) \cong k$ and $H^{-1}(C_i^\bullet) \cong A$ and each is determined by a class in $\mathrm{Ext}^2(k, A)$ given by

$$\begin{aligned} \alpha_1 &: \quad 0 \to A \to A^{\oplus 2} \to A \to k \to 0 \\ \alpha_2 &: \quad 0 \to A \xrightarrow{\mathrm{id}} A \xrightarrow{0} k \xrightarrow{\mathrm{id}} k \to 0 \end{aligned}$$

Geometrically speaking, $A = \mathcal{O}_{\mathbb{A}^2}$ and $k = \mathcal{O}_{(0,0)}$ is the structure sheaf of the point $(0,0) \in \mathbb{A}^2$. Two-step extensions can be computed using Serre Duality

$$\mathrm{Ext}^2_{\mathbb{A}^2}(\mathcal{O}_{(0,0)}, \mathcal{O}_{\mathbb{A}^2}) \cong \mathrm{Ext}^0_{\mathbb{A}^2}(\mathcal{O}_{\mathbb{A}^2}, \mathcal{O}_{(0,0)})^\vee \cong k$$

It can be shown that α_2 is trivial while α_1 is not[1]. This implies $C_1^\bullet \overset{q}{\ncong} C_2^\bullet$.

Detour: Let $C^\bullet$ be a complex and assume $H^n(C^\bullet) = 0$ for all $n > N$. We define the right-truncated complex $\tau_{\leq N} C^\bullet$ to be

$$\cdots \to C^{N-2} \to C^{N-1} \to Z^N \to 0 \to \cdots$$

We have a map of complexes $\tau_{\leq N} C^\bullet \to C^\bullet$ which induces an isomorphism on the level of cohomology. Similarly, when $H^m(C^\bullet) = 0$ for all $m < M$ we can define the left-truncated complex $\tau_{\geq M} C^\bullet$. We have a map $C^\bullet \to \tau_{\geq M} C^\bullet$ which induces isomorphism on the level of cohomology. In addition, we have maps of complexes $\tau_{\leq N} C^\bullet \to H^N(C^\bullet)[-N]$ and $H^M(C^\bullet)[-M] \to \tau_{\geq M} C^\bullet$.

Let $C^\bullet$ be a complex with two adjacent non-vanishing cohomology groups, say in degrees -1 and 0 and consider $f : H^{-1}(C^\bullet)[1] \to \tau_{\geq -1} C^\bullet$ as above. Taking the cone of f we have an exact triangle in the homotopy category

$$\begin{array}{ccc} H^{-1}(C^\bullet)[1] & \rightarrow & \tau_{\geq -1} C^\bullet \\ & \nwarrow\!\!\!\!\!\!\times \qquad \swarrow & \\ & \mathrm{Cone(f)} & \end{array}$$

[1] α_1 is the Kozul resolution of k over A.

Explicitly, the cone of f is the complex

$$0 \to H^{-1}(C^\bullet) \to C^{-1}/\mathrm{Im}d^{-2} \to C^0 \to C^1 \to \cdots$$

In particular, we have a map $H^0(C^\bullet)[0] \to \mathrm{Cone(f)}$ which induces an isomorphism on cohomology. In the derived category we have an exact triangle

$$\begin{array}{ccccc} H^{-1}(C^\bullet)[1] & & \to & & C^\bullet \\ & \not\nwarrow & & \swarrow & \\ & & H^0(C^\bullet)[0] & & \end{array}$$

The morphism $H^0(C^\bullet)[0] \not\to H^{-1}(C^\bullet)[1]$ gives a class in $\mathrm{Ext}^2(H^0(C^\bullet), H^{-1}(C^\bullet))$.

Moral of Story: Showing that two complexes are not quasi-isomorphic when they have isomorphic cohomology groups requires the use of more subtle invariants and the one we just used cannot be pushed further to a more general setting. An $\mathcal{A}_\infty$ structure, on the other hand, provides a complete set of invariants.

Claim: If we regard A as an $\mathcal{A}_\infty-$algebra and $C_1^\bullet, C_2^\bullet$ as $\mathcal{A}_\infty-$modules over A then $C_1^\bullet \overset{q}{\cong} C_2^\bullet$ iff they are isomorphic as $\mathcal{A}_\infty-$modules.

1.2 Kontsevich's Motivation.

Theorem: [Bondal-van den Bergh; Toën - Vaquie] Let X be a complex manifold. Then X is algebraic iff $D^b(X)$ admits a split generator.

Fact: If E is a split generator for $D^b(X)$ then the derived category can be recovered from the $\mathcal{A}_\infty-$algebra $\mathrm{Ext}^\bullet(E, E)$.

The analogy to keep in mind is: affine varieties to commutative rings are like algebraic varieties to $\mathcal{A}_\infty-$rings. In the former case the correspondence is unique, in the later case it is not. $D^b(X)$ can have several split generators and X can not be fully recovered from $D^b(X)$. Still, we think of any algebraic variety as "affine" in the $\mathcal{A}_\infty$ world.

1.3 Stasheff's Motivation.

Let X be a topological space with a base point $*$ and consider the space $\Omega(X, *)$ of loops based at $*$, that is, the space of maps $\gamma : [0, 1] \to X$ such that $\gamma(0) = * = \gamma(1)$. We wish to consider $\Omega(X, *)$ as a group with multiplication given by concatenating two loops and tracing each at twice the speed.

This multiplication is clearly not associative; the maps $(\gamma_1 \circ \gamma_2) \circ \gamma_3$ and $\gamma_1 \circ (\gamma_2 \circ \gamma_3)$ are not identical. But they are homotopic, and we can prescribe a

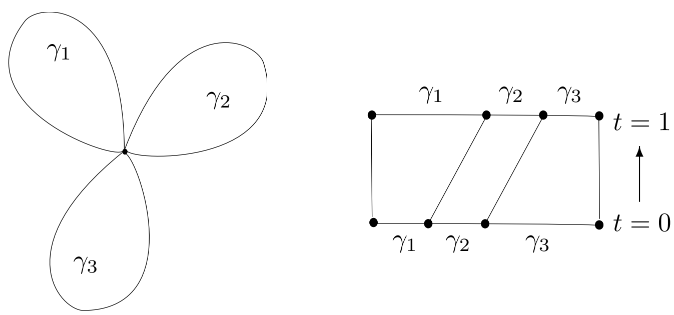


Figure 1: Composition of loops.

homotopy $h^{(3)}$ between them independent of γ_i. See for example fig. 1. Given four loops, we can homotope $((\gamma_1 \circ \gamma_2) \circ \gamma_3) \circ \gamma_4$ to $\gamma_1 \circ (\gamma_2 \circ (\gamma_3 \circ \gamma_4))$ using $h^{(3)}$ in more than one way, and we can prescribe a homotopy $h^{(4)}$ between those, which is again independent of γ_i. And so on and so forth, every homotopy we prescribe will satisfy higher homotopy associativity relations. This collection of operations $m, h^{(3)}, h^{(4)}, \ldots$ characterizes $\Omega(X, *)$ up to homotopy equivalence.

Theorem: [Stasheff] $Y \overset{h}{\cong} \Omega(X, *)$ iff Y can be equipped with such collection of operations.

An $\mathcal{A}_\infty-$algebra is the algebraic analogue of this structure.

1.4 $\mathcal{A}_\infty$ Algebras.

Almost Definition: A (flat) $\mathcal{A}_\infty$ algebra is a vector space together with operations

$$m_k : A^{\otimes k} \to A \quad , \quad k \geq 1$$

satisfying an infinite sequence of equations

$$\mathrm{Assoc}_n : \sum \pm m_a(x_1, \ldots, x_l, m_b(x_{l+1}, \ldots, x_{l+b}), x_{l+b+1}, \ldots, x_n) = 0$$

where we sum over all $a + b = n + 1$ and $l < a$. On the level of an "almost" definition we ignore the issue of signs. We can represent each element in the sum by a tree with n leaves, one root and one internal edge.
We write down the first few associativity equations.

$$\begin{aligned}
\mathrm{Assoc}_1 \quad &: \quad m_1(m_1(x_0)) = 0 \\
\mathrm{Assoc}_2 \quad &: \quad m_1(m_2(x, y)) \pm m_2(m_1(x), y) \pm m_2(x, m_1(y)) = 0 \\
\mathrm{Assoc}_3 \quad &: \quad m_2(x, m_2(y, z)) \pm m_2(m_2(x, y), z) \pm m_1(m_3(x, y, z)) \\
&\qquad \pm m_3(m_1(x), y, z) \pm m_3(x, m_1(y), z) \pm m_3(x, y, m_1(z)) = 0
\end{aligned}$$

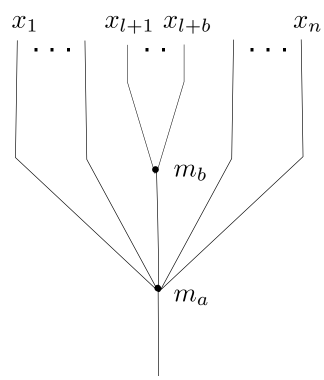


Figure 2: Tree diagram.

Remarks:
• m_1 is a differential on A so we will often denote $d := m_1$ (Assoc_1 is just $d^2 = 0$). Also, we will often write xy for $m_2(x, y)$.
• Assoc_2 is the Leibnitz Rule $d(xy) = (dx)y + x(dy)$.
• By Assoc_3 we have

$$(xy)z - x(yz) = dm_3(x, y, 3) \pm m_3(dx, y, z) \pm m_3(x, dy, z) \pm m_3(x, y, dz)$$

This means m_2 descends to an associative multiplication on $H^\bullet(A, d)$.

Immidiate Examples:
• Any associative algebra is an $\mathcal{A}_\infty$ algebra with m_2 the only non-vanishing operation.
• A dg-algebra is an $\mathcal{A}_\infty$ algebra with $m_k = 0$ for all $k \geq 3$.

Interesting Example: Let A be an associative algebra. Let α be a Hochschild k-cochain, meaning a linear map $\alpha : A^k \to A$, and $k \geq 3$. Assume α is closed. This means

$$x_1\alpha(x_2, \ldots, x_{k+1}) - \alpha(x_1x_2, x_3, \ldots, x_{k+1}) \\ + \alpha(x_1, x_2x_3, \ldots, x_{k+1}) - \ldots \pm \alpha(x_1, \ldots, x_k)x_{k+1} = 0$$

We construct an $\mathcal{A}_\infty$ algebra, A_α, with the same underlying vector space and $m_2 = \cdot$, $m_k = \alpha$. The fact that α is closed implies Assoc_k (all other associativity equations are obviously satisfied).

Curved $\mathcal{A}_\infty$ Algebras: To get a *curved* $\mathcal{A}_\infty$ algebra we add a map

$$m_0 : k \to A$$

Let $W := m_0(1)$. We have an additional associativity constraint

$$\text{Assoc}_0 : dW = 0$$

The other associativity constraints need to be readjusted. In particular

$$\text{Assoc}_1 : d^2(x) \pm xW \pm Wx = 0$$

This means we no longer have a differential. An immediate example is given by an associative algebra A with an element $W \in Z(A)$ and multiplication as before.

Grading: There are two types of graded $\mathcal{A}_\infty$ algebras that appear in the literature. One is a $\mathbb{Z}$-graded and the other is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. The operations are defined to be

$$m_k : A^{\otimes k} \to A[2-k]$$

Note that m_2 does not shift. The differential shifts by 1, as expected.

1.5 Co-free Co-Algebra.

Let V be a vector space. Consider the free algebra on V to be $TV = \oplus_{k\geq 0} V^{\otimes k}$ with multiplication given by

$$(x_1|\cdots|x_k)\cdot(x_{k+1}|\cdots|x_n) = (x_1|\cdots|x_n)$$

We also have a co-free co-algebra on V. It is given by the same underlying vector space $CV = \oplus_{k\geq 0} V^{\otimes k}$ with co-multiplication given by

$$\Delta(x_1|\cdots|x_n) = \sum_{k=0}^{n} (x_1|\cdots|x_k) \otimes (x_{k+1}|\cdots|x_n)$$

Note this does not make the underlying vector space into a bi-algebra[2]. For $x \in V^{\otimes 1}$ and $y \in V^{\otimes 1}$ we have

$$\begin{aligned} \Delta(x|y) &= 1 \otimes (x|y) + x \otimes y + (x|y) \otimes 1 \\ \Delta(x)\Delta(y) &= (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) \\ &= 1 \otimes (x|y) + 2x \otimes y + (x|y) \otimes 1 \end{aligned}$$

which shows Δ is not a morphism of algebras.

[2] There are other products / co-products called the *shuffle product / co-product* which make $\oplus_{k\geq 0} V^{\otimes k}$ into a bi-algebra.

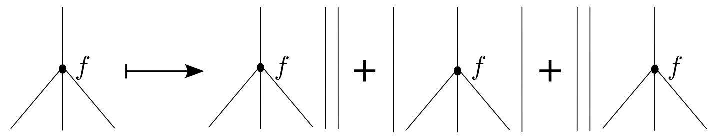


Figure 3: Defining $\Box_f$.

Theorem: There exists a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Vect}}(V, TV) \xrightarrow{\sim} \mathrm{Der}(\mathrm{TV}, \mathrm{TV})$$

where Der stands for derivations.

This natural isomorphism is given by sending $f : V \to TV$ to a derivation

$$\Box_f : (x_1|\cdots|x_n) \mapsto \sum_k (x_1|\cdots|f(x_k)|\cdots|x_n)$$

For example, a map $f : V \to V^{\otimes 3}$ can represented as a tree with one branch and three roots. The corresponding $\Box_f$ will be evaluated on $V^{\otimes 3}$ and take its values in $V^{\otimes 5}$ as in fig. 3.

Checking $\Box_f$ is a derivation means showing the following diagram commutes.

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\cdot} & A \\ \Box_f \otimes 1 + 1 \otimes \Box_f \downarrow & & \downarrow \Box_f \\ A & \xrightarrow{\Box_f} & A \end{array} \tag{1}$$

Theorem: There exists a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Vect}}(CV, V) \xrightarrow{\sim} \mathrm{CoDer}(\mathrm{CV}, \mathrm{CV})$$

where CoDer stands for co-derivations.

Co-derivations can be defined by inverting all arrows in equation 1 and replacing multiplication by co-multiplication. To define the natural isomorphism in this case we just turn fig. 3 up-side down.

Theorem: An $\mathcal{A}_\infty$ structure on V is a square-zero co-derivation on CV.

Given a co-derivation $\Box$ on CV we get a map $f : CV \to V$ which translates into a collection of operations $f^{(k)} : V^{\otimes k} \to V$. The requirement that $\Box^2 = 0$

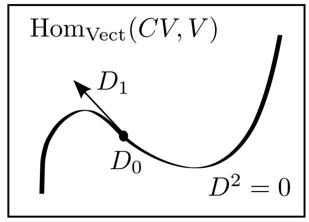


Figure 4: Deformation of $\mathcal{A}_\infty$ structure.

translates into the Assoc_n equations. Taking $\square^2$ means we are considering trees with exactly one internal edge.

Deformations of $\mathcal{A}_\infty$ Algebras: Given the above theorem, we can think of the 'space' of $\mathcal{A}_\infty$ structures on an underlying vector space V as a 'subspace' of $\mathrm{Hom}_{\mathrm{Vect}}(CV, V) \cong \mathrm{Hom}_{\mathrm{CoDer}}(CV, CV)$ given by the locus of points satisfying $D^2 = 0$.

We wish to understand "$T_A(D^2 = 0)$" – the tangent space to $\{D^2 = 0\}$ at a square-zero derivation D_0 corresponding to an $\mathcal{A}_\infty$−algebra A. A first order deformation of D_0 is of the form $D_0 + \hbar D_1$ satisfying $(D_0 + \hbar D_1)^2 = 0$ up to first order in $\hbar$. This implies the tangent space is given by

$$T_A(D^2 = 0) = \{D_1 \in \mathrm{Hom}_{\mathrm{CoDer}}(CV, CV) \,|\, D_0 D_1 + D_1 D_0 = 0\}$$

This is equivalent to the Hochschild complex of the $\mathcal{A}_\infty$ algebra A.

2 Lecture 2

2.1 Derived Categories.

Motivation: Consider a simply connected manifold X with its de-Rham complex $(\Omega(X), d)$. Knowing the complex with a bit of extra data completely determines the rational homotopy type of X. On the other hand, just knowing $H^\bullet_{DR}(X)$ is not enough. The moral of this example is that working with chain complexes, rather than with their cohomology groups, retains more information.

Categories of Complexes: Let $\mathcal{A}$ be an abelian category. This could be the category of modules over a ring or the category of coherent sheaves over a scheme. We construct the category $\mathrm{dg}(\mathcal{A})$. Its objects are complexes of objects in $\mathcal{A}$. Its Hom sets are differential graded vector spaces defined by

$$\mathrm{Hom}^i_{\mathrm{dg}(\mathcal{A})}(C^\bullet, D^\bullet) := \prod_j \mathrm{Hom}_{\mathcal{A}}(C^j, D^{j+i})$$

$$df^\bullet := d_{D^\bullet} \circ f - f \circ d_{C^\bullet}$$

$\mathrm{dg}(\mathcal{A})$ is a dg-category, meaning it is a category enriched over dgVect .

Homotopy Category: We define the *homotopy category* $H_o(\mathcal{A})$ to be the category whose objects are the same as those of $\mathrm{dg}(\mathcal{A})$ and

$$\mathrm{Hom}_{H_o(\mathcal{A})}(C^\bullet, D^\bullet) := H^0(\mathrm{Hom}_{\mathrm{dg}(\mathcal{A})}(C^\bullet, D^\bullet))$$

To understand what the morphisms of $H_o(\mathcal{A})$ are take a 0-cocycle $f^\bullet$ in $\mathrm{Hom}^0(C^\bullet, D^\bullet)$. It is a map $f^\bullet : C^\bullet \to D^\bullet$ such that $df^\bullet = d_{D^\bullet} f^\bullet - f^\bullet d_{C^\bullet} = 0$, namely, $f^\bullet$ is a chain map. Modding out by 0-coboundaries amounts to modding out by homotopy. Hence $H_o(\mathcal{A})$ is the category whose objects are chain complexes of objects in $\mathcal{A}$ and whose morphisms are homotopy classes of chain maps. Note that

$$H^n(\mathrm{Hom}_{\mathrm{dg}(\mathcal{A})}(C^\bullet, D^\bullet)) = H^0(\mathrm{Hom}_{\mathrm{dg}(\mathcal{A})}(C^\bullet, D^\bullet[n]))$$

therefore knowing the shift functor allows us to compute all cohomology groups of Hom spaces in $\mathrm{dg}(\mathcal{A})$.

Definition: A morphism in $H_o(\mathcal{A})$ is called a *quasi-isomorphism* if $H^\bullet(f^\bullet)$ is an isomorphism.

Example: Let $\mathcal{A} = \mathbb{Z}$-mod. Consider the following chain map.

$$\begin{array}{ccccccc} 0 & \to & \mathbb{Z} & \stackrel{\times 2}{\to} & \mathbb{Z} & \to & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & \mathbb{Z}/2\mathbb{Z} & \to & 0 \end{array}$$

The map induces an isomorphism in cohomology. But there is no chain map going in the other direction. The map is a quasi-isomorphism but is not invertible in $H_o(\mathcal{A})$. We will construct a category of chain complexes in which this map is invertible.

Localization: Let $\mathcal{C}$ be a category and S a collection of morphisms in $\mathcal{C}$). $S^{-1}\mathcal{C}$ is a category together with a functor $\mathcal{C} \to S^{-1}\mathcal{C}$ such that the following holds.

- The image of any morphism in S is an isomorphism in $S^{-1}\mathcal{C}$
- $S^{-1}\mathcal{C}$ satisfies the following universal property: for any other category $\mathcal{D}$ and a functor $\mathcal{C} \to \mathcal{D}$ with the above property, there exists a unique functor $S^{-1}\mathcal{C} \to \mathcal{D}$ making the appropriate diagram commute.

We can construct $S^{-1}\mathcal{C}$ explicitly. Its object are the same as those of $\mathcal{C}$. Morphisms from A to B are given by 'zig-zag' diagrams.

$$\begin{array}{ccccccccccccc} & X_1 & & & X_2 & & & \cdots & & X_n & & \\ \swarrow & & \searrow & \swarrow & & \searrow & & & \swarrow & & \searrow & \\ A & & & Y_1 & & & Y_2 & \cdots & Y_{n-1} & & & B \end{array}$$

where arrows pointing left are in S (the ones pointing right are arbitrary).

The Derived Category: Consider the homotopy category $H_o(\mathcal{A})$ and take S to be the collection of all quasi-isomorphisms in $H_o(\mathcal{A})$. We define the *derived category* of $\mathcal{A}$ to be $\mathcal{D}(\mathcal{A}) := S^{-1}H_o(\mathcal{A})$.

Both $H_o(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ are additive categories, but they are no longer abelian, for example, we added new morphisms to $\mathcal{D}(A)$ but not necessarily their kernels and co-kernels.

Triangulated Category: Both $H_o(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ are triangulated categories. Roughly speaking, a *triangulated category* is a category equipped with an auto-equivalence, a 'shift', denoted by "[1]", and a distinguished collection of diagrams called *distinguished triangles* of the form $A \to B \to C \to A[1]$ satisfying some axioms. We often write down distinguished triangles as

$$\begin{array}{ccccc} A & & \to & & B \\ & \not\nwarrow & & \swarrow & \\ & & C & & \end{array}$$

For any morphism $A \xrightarrow{f} B$ there exists a distinguished triangle $A \xrightarrow{f} B \to C \to A[1]$ (C need not be unique), in particular, $A \xrightarrow{\text{id}} A \to 0 \to A[1]$ is a distinguished triangle. Given a triplet of isomorphisms making all rectangles commute

$$\begin{array}{ccccccc} A & \to & B & \to & C & \to & A[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \to & B' & \to & C' & \to & A'[1] \end{array}$$

the bottom line is also a distinguished triangle.

For any morphism class $C^\bullet \xrightarrow{f^\bullet} D^\bullet$ in $H_o(\mathcal{A})$, we can define the *cone of* $f^\bullet$ to be a complex $(\text{Cone} f^\bullet)^i = C^{i+1} \oplus D^i$ with differential

$$d^i_{\text{Cone}} = \begin{pmatrix} d_C^{i+1} & 0 \\ f^{i+1} & d_D^i \end{pmatrix}$$

There is a distinguished triangle in $H_o(\mathcal{A})$ given by inclusion / projection onto each summand

$$\begin{array}{ccccc} C^\bullet & & \xrightarrow{f^\bullet} & & D^\bullet \\ & \not\nwarrow & & \swarrow & \\ & & \text{Cone}(\text{f}^\bullet) & & \end{array}$$

We think of $\text{Cone}(f^\bullet)$ as encoding the kernel and co-kernel of $f^\bullet$.

Given a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ we have a distinguished triangle in $\mathcal{D}(\mathcal{A})$

$$\begin{array}{ccccc} A[0] & & \to & & B[0] \\ & \not\nwarrow & & \swarrow & \\ & & C[0] & & \end{array}$$

arising from

$$\begin{array}{ccccccc} 0 & \to & 0 & \to & C & \to & 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \to & A & \to & B & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & A & \to & 0 & \to & 0 \end{array}$$

where the top arrow is a quasi-isomorphism. The above triangle is *not* a distinguished triangle in $H_o(\mathcal{A})$.

Enhancement: We wish to argue that $\mathcal{D}(\mathcal{A})$ has an *enhancement*, namely, that there exists a dg-category $\mathrm{DG}(\mathcal{A})$ such that $H_o(\mathrm{DG}(\mathcal{A})) \cong \mathcal{D}(\mathcal{A})$.

Theorem: If $C^\bullet$ and $D^\bullet$ are complexes of injectives in $\mathcal{A}$, then a chain map $C^\bullet \to D^\bullet$ is a quasi-isomorphism if and only if it is invertible up to homotopy.

We take $\mathrm{DG}(\mathcal{A})$ to be $\mathrm{dg}(\mathrm{Inj}(\mathcal{A}))$.

2.2 Homological Perturbation Lemma.

Let $A^\bullet$ be a (flat) $\mathcal{A}_\infty$ algebra ($A^\bullet$ is $\mathbb{Z}$ -graded). We denote $d := m_1$ and consider the complex $(A^\bullet, d)$. The problem we are trying to solve is decomposing $A^\bullet = B^\bullet \oplus$ (something), such that $B^\bullet$ is a smaller complex encoding the cohomology of $A^\bullet$. The motivation for this arises from Hodge theory where the complex of harmonic forms is a subcomplex of $A^{p,q}(X)$ and encodes its cohomology, namely, there is a unique harmonic form in every class.

From now on we will drop bullets from our notation; all objects and maps are in the category of complexes. Let $\pi : A \to A$ be a chain map such that $\pi^2 = \pi$. This decomposes A into $A = \mathrm{Im}\pi \oplus \ker \pi$. We denote $B = \mathrm{Im}\pi$.

Assume π is homotopy equivalent to the identity, namely, there exists $H : A \to A[1]$ such that $\mathrm{id} - \pi = dH + Hd$. Denote the inclusion $i : B \to A$, the projection $p = i \circ \pi$, and we have $\pi = i \circ p$.

We will define an $\mathcal{A}_\infty$ structure on B. The differential d^B will be given by the restriction of d^A to B. The multiplication will be given by $m_2^B := p \circ m_2^A \circ (i \otimes i)$.

To define higher multiplication, consider T a planar tree with n leaves and one root. All internal vertices are of valence at least 3. Define $m_n^T : B^{\otimes n} \to B[2-n]$ by labeling all the leaves of T by the embedding i, the root of T by the projection p, all internal vertices of T by m_k^A (for the appropriate k), and internal edges by the homotopy H. An example of such labeling appears in fig. 5.

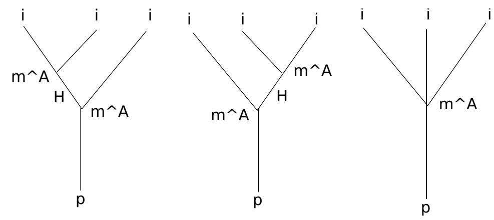


Figure 5: Labeled trees m_3^T.

Define the higher multiplication on B to be

$$m_n^B := \sum_T \pm m_n^T$$

Note that m_n^B has the right degree. Let V_T be the set of vertices of T, let E_T be the set of edges and E_T^{int} its subset of internal edges, then

$$\begin{aligned}
\deg(m_n^T) &= -|E_T^{\text{int}}| + \textstyle\sum_{v \in V_T} 2 - (\deg(v) - 1) \\
&= -|E_T^{\text{int}}| + 3|V_T| - (2|E_T^{\text{int}}| + n + 1) \\
&= 3(|V_T| - |E_T^{\text{int}}|) - n - 1 \\
&= 2 - n
\end{aligned}$$

Theorem: d^B, m_n^B form a $\mathcal{A}_\infty$ structure on B.

Proof: The proof will be entirely combinatorial. We will demonstrate it for m_3^B. We will ignore signs completely (signs should work out the way we expect them to). Remember we need to show $\text{Assoc}_3^B = 0$. To make things clearer, we will denote all operations in A by bullets, all H's by circles, and all π's by stars. $\text{Assoc}_3(B)$ is given in fig. 6 (operations in B are not marked). We expand

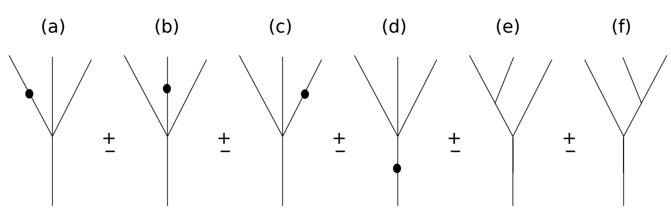


Figure 6: Assoc_3^B.

$\text{Assoc}_3(B)$ in terms of operations in A, π and H (see fig 7). We use the fact that π is homotopy equivalent to the identity. Pictorially we can represent it as in fig. 8. We plug in the homotopy into $\text{Assoc}_3(B)$. This introduces three terms

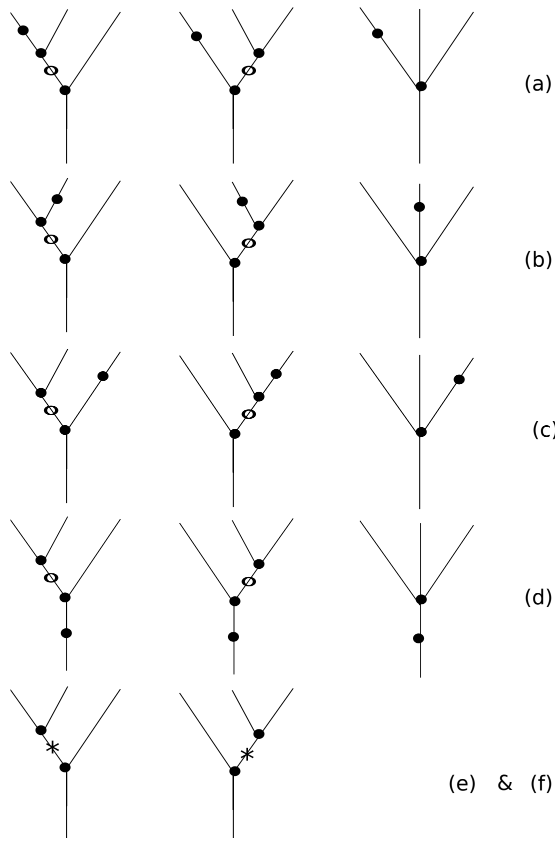


Figure 7: Expanding Assoc_3^B in terms of m_n^A,π and H.

replacing each (e),(f) in fig. 7. We find that the terms in the right most column of fig. 7 and a couple of terms from (e),(f) cancel out thanks to $\text{Assoc}_3(A)$. We are left with trees appearing in fig. 9. Note that now each black dot on a leaf or a root is closest to one distinguished vertex. We re-group all trees according to such common distinguished vertex. But now, each row in fig. 9 vanishes due to $\text{Assoc}_2(A)$. That finishes the proof of the vanishing of $\text{Assoc}_3(B)$. Similar considerations work for higher order Assoc.

Theorem: A and B are quasi-isomorphic as $\mathcal{A}_\infty$ algebras[3]. In particular, if we take $B := H^\bullet(A)$ and choose a (not canonical) splitting $A = B \oplus (\text{something})$, then A is quasi-isomorphic to $H^\bullet(A)$.

[3] A map of $\mathcal{A}_\infty$ algebras is more than a map of the underlying $\mathbb{Z}$-graded vector spaces. A map of $\mathcal{A}_\infty$ algebras $B \to A$ corresponds to a map of co-algebras $CB \to CA$. This map is completely determined by a collection of maps $B^{\otimes n} \to A[-]$ (satisfying some conditions).

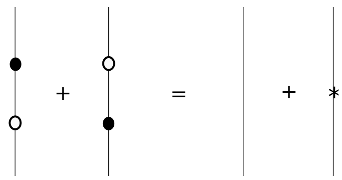


Figure 8: Homotopy.

Application: Given a space X we can construct a "big" dg-category DG(X) associated to it. We can then use the homological perturbation lemma to replace DG(X) by a "small" $\mathcal{A}_\infty$ category, $\mathcal{A}_\infty(X)$. We pay the price of adding higher order multiplications.

Theorem: If E is a split generator of $D^b(X) := D^b(\mathrm{Coh(X)})$ then the $\mathcal{A}_\infty$ algebra $\mathrm{End}_{D^b(X)}(E)$ carries all information of $\mathcal{A}_\infty(X)$.

Split Generators: Roughly speaking, an object E in a triangulated category is a *split generator* if applying shifts, cones and direct summands to it gives everything.

Example: A result due to Beilinson states that $\mathcal{O}, \mathcal{O}[1], \ldots, \mathcal{O}[n]$ generate $D^b(\mathbb{P}^n)$. This is no longer true for an elliptic curve E. To see why, consider the surjection

$$D^b(E) \to K^0(E)$$
$$C^\bullet \to \sum_i (-1)^i [C^i]$$

This map has the following properties: a distinguished triangle $A \to B \to C \to A[1]$ maps to $[A] - [B] + [C] = 0$ and a shift $A[-1]$ maps to $-[A]$. If a finite set of objects was to generate $D^b(E)$ their images would then generate $K^0(E)$. But $K^0(E)$ is not finitely generated.

On the other hand, we have a double cover $\pi : E \to \mathbb{P}^1$ (branched over four points). We can use the fact that $\pi_*(C)$ is generated by $\mathcal{O}, \mathcal{O}[1]$ in $D^b(\mathbb{P}^1)$, and the fact that any $C \in D^b(E)$ is a direct summand of $\pi^*\pi_*(C)$, to argue that $\pi^*\mathcal{O}, \pi^*\mathcal{O}[1]$ split generate $D^b(E)$.

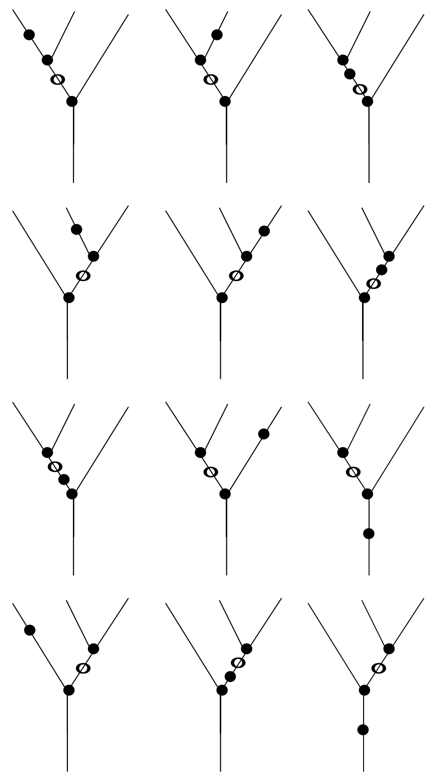


Figure 9: Using homotopy and re-grouping.

3 Lecture 3

3.1 Operads.

Definition: An *operad* O in a monoidal category $\mathcal{C}$ is a collection of objects $O(n) \in \mathcal{C}$ for all $n \geq 1$, that carry a Σ_n action[4], together with composition operators

$$\circ : O(k) \otimes O(n_1) \otimes \cdots \otimes O(n_k) \to O(n_1 + \ldots + n_k)$$

satisfying some axioms (associativity and identity). A map of two operads is a collection of morphisms that preserve the structure.

Associative Algebras: Denote the multiplication of an associative algebra by a binary tree with two inputs and one output. Multiplication of n elements can be represented by a binary tree with n inputs and one output. This representation depends on the order in which we choose to multiply.

[4] Σ_n is the permutation group on n letters.

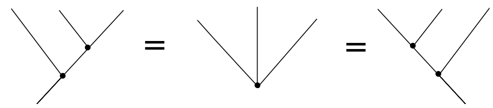


Figure 10: Associativity in trees.

For example, we have two binary trees with three inputs. The fact that multiplication is associative tells us that the two trees are one and the same. This identity is expressed in fig. 10.

The operad describing associative algebras is an operad in Sets. Each set $O(n)$ contains trees with n inputs, one output, and one vertex of valence $n+1$, whose leaves are labeled by $\{1,\dots,n\}$ (Σ_n acts on the labels). We denote this operad by Ass.

The operad describing commutative algebras is also an operad in Sets with $O(n) = \{*\}$ (the action of Σ_n on $O(n)$ is trivial). We can think of $O(n)$ as containing one tree with n inputs as above and no labeling. We denote this operad by Com.

Another important example arises from objects in monoidal categories. Let V be such object. Define an operad $\mathrm{End}(V)$ to be

$$\mathrm{End}(V)(n) := \mathrm{Hom}_{\mathcal{C}}(V^{\otimes n}, V)$$

with composition given by

$$f, f_1, \dots, f_k \mapsto f \circ f_1 \otimes \cdots \otimes f_k$$

Definition: Let O be an operad in $\mathcal{C}$. An *algebra over* O is an object V in a category enriched over $\mathcal{C}$ equipped with a map of operads $O \to \mathrm{End}(V)$.

An associative algebra is an algebra over Ass and a commutative algebra is an algebra over Com. The advantage of this terminology is that it allows us to talk about associative or commutative algebras in Sets, topological spaces and so forth.

More Examples: The *free operad* with one multiplication is an operad in Sets with

$$\mathrm{Free}(n) := \{\text{binary trees with } n \text{ leaves and one root}\}$$

Unlike in Ass, here we do distinguish between different orderings of multiplication, for example, see fig. 11.

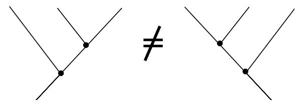


Figure 11: Distinguished orderings of multiplication.

The *Lie operad*, Lie, is given as a quotient of the free operad by the anti-commutativity and Jacobi identity relations. To make sense of these relations, we take Free to mean the vector space spanned by all binary trees with n leaves and one root. Hence Lie is an operad in vector spaces.

The Little Discs Operad: We define an operad D_k in topological spaces with $D_k(n)$ being space consisting of n k-dimensional discs embedded in the unit k-dimensional disc $\mathbb{D}^k$ (see fig. 12). Composition maps are given by rescaling discs and sticking them one into the other.

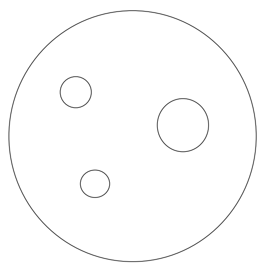


Figure 12: A point in $D_2(3)$.

If we think of each little disc as determined by its center and radius we can embed

$$D_k(n) \subset \underbrace{\mathbb{D}^k \times \cdots \times \mathbb{D}^k}_{n} \times \underbrace{(0,\infty) \times \cdots \times (0,\infty)}_{n}$$

where discs are not allowed to intersect. Up to homotopy, $D_k(n)$ is the configuration space of n distinct points in $\mathbb{D}^k$

$$D_k(n) \cong_h \underbrace{\mathbb{D}^k \times \cdots \times \mathbb{D}^k}_{n} \backslash \{\text{all diagonals}\}$$

Therefore $\pi_1(D_2(n)) = B_n$ (the braid group on n strands).

Passing to Homology: Given any operad O in topological spaces we can construct a new operad $H_*(O)$ in graded vector spaces or $C_*(O)$ in differential graded vector spaces by taking homology or chains of each $O(n)$ respectively.

Let us check what $H_*(D_1)$ should be. $H_*(D_1(n))$ is concentrated in degree 0 for all n and is a vector space of dimension $n!$. It is not hard to guess that $H_*(D_1) = \mathrm{Ass}$ when considering Ass as an operad over vector spaces by taking the vector space generated by $\mathrm{Ass}(n)$ in each degree. If instead we take chains then $C_*(D_1)$ is the $\mathcal{A}_\infty$ operad.

Let us go one dimension up. For $n = 1$, $H_*(D_2(1)) = H_*(\mathbb{D}_2) = k[0]$. This means an algebra over $H_*(D_2)$ carries a unary operation in degree 0. Now consider $n = 2$. To understand what $D_2(2)$ looks like note that any two points in $\mathbb{D}^2$ determine a line intersecting the boundary $\partial\mathbb{D}$. We claim $H_*(D_2(2)) \cong H_*(S^1) \cong k[0] \oplus k[1]$. This means an algebra over $H_*(D_2)$ carries two binary operations, one in degree 0 (multiplication) and the other in degree 1 (bracket). Note $D_2(n)$ is connected for all n. This implies our multiplication is associative and commutative. The bracket is in fact a Lie bracket and is compatible with multiplication in the sense that $[xy, z] = x[y, z] + y[x, z]$.

Gerstenhaber Algebra: A *Gerstenhaber algebra* (G-algebra) is a graded vector space, A, equipped with associative commutative multiplication of degree 0 and a Lie bracket of degree -1 satisfying $[xy, z] = x[y, z] + y[x, z]$. A Gerstenhaber algebra is an algebra over $H_*(D_2)$.

Example: Let X be a smooth manifold. Let $A = \oplus_i \Gamma(\Lambda^i T_X)$ be the algebra of poly-vector fields. A is a G-algebra. The multiplication is given by the wedge product and the bracket is the Lie bracket extended to A via $[v_1 \wedge v_2, v_3] = v_1 \wedge [v_2, v_3] + v_2 \wedge [v_1, v_3]$.

Deligne Conjecture: Let A be an associative algebra, then $HH^*(A)$ is a Gerstenhaber algebra. In other words, if A an algebra over $H_*(D_1)$ then $HH^*(A)$ is an algebra over $H_*(D_2)$.

In fact, a more general statement is true.

Theorem:[Kontsevich - Lurie] If A is an algebra over $H_*(D_n)$, then $HH^*(A)$ is an algebra over $H_*(D_{n+1})$. This statement is also true at chain level, namely, if A is an algebra over $C_*(D_n)$, then $C^*(A)$ is an algebra over $C_*(D_{n+1})$, where $C^*(A)$ is the Hochschild cochain complex of A, appropriately defined.

3.2 Hochschild Cohomology.

We already encountered Hochschild cohomology in section 1.4 where we constructed an example of an $\mathcal{A}_\infty$−algebra using a Hochschild cocycle.

Let A be an associative algebra over a field k. We define the vector space of p-chains to be

$$C^p(A) := \mathrm{Hom}_k(A^{\otimes p}, A)$$

We define a differential $b : C^p(A) \to C^{p+1}$ to be

$$\begin{aligned}(bf)(a_1,\ldots,a_{p+1}) &= a_1 \cdot f(a_2,\ldots,a_{p+1}) - f(a_1 \cdot a_2,\ldots,a_{p+1}) + \cdots \\ &\quad \cdots \pm f(a_1,\ldots,a_p) \cdot a_{p+1}\end{aligned}$$

We claim $(C^\bullet(A), b)$ is a chain complex. We denote $HH^\bullet(A) := \ker b/\mathrm{Im} b$.

Low Degree Comology: Let us consider the first couple of terms in $HH^\bullet(A)$. Let $a \in \mathrm{Hom}_k(k, A)$ be a 0-cochain. We identify it with $a(1)$ and $b(a) : x \mapsto xa - ax$. A 0-cocycle is therefore an element of the center and $HH^0(A) = Z(A)$. Let $f \in \mathrm{Hom}_k(A, A)$ be a 1-cochain and $(bf)(x, y) = xf(y) - f(xy) + f(x)y$. A 1-cocycle is therefore a derivation of A. A 1-coboundary is given by the natural map $[\ ,\ ] : A \to \mathrm{Der}(A)$ taking a to $[-, a]$. We have $HH^1(A) = \mathrm{Der}(A)/[-, A]$.

When A is commutative $HH^0(A) = A$ and $HH^1(A) = \mathrm{Der}(A)$. A prototypical example is $A = C^\infty(X)$ and $HH^0(A) = A = \Gamma(\Lambda^0 T_X)$, $HH^1(A) = \mathrm{Der}(A) = \Gamma(\Lambda^1 T_X)$. In higher degrees we have $HH^m(A) = \Gamma(\Lambda^m T_X)$. We already argued that in this example $HH^\bullet(A)$ has a structure of a G-algebra.

Deformations: Given an algebra A, we wish to consider a family of algebras $A_\hbar$, with the same underlying vector space, such that $A \cong A_0$. In other words, we wish to deform the multiplication on A to

$$x * y = xy + \hbar c_2(x, y) + \hbar^2 c_3(x, y) + \ldots$$

Up to first order, the associativity of $x * y$ is equivalent to $dc_2 = 0$. It is not hard to show that deformations via 2-coboundaries are trivial up to first order. In other words, $HH^2(A)$ controls deformations of the multiplication on A up to first order.

Multiplication: We define multiplication in $C^\bullet(A)$

$$\begin{aligned}&C^p(A) \otimes C^q(A) \to C^{p+q}(A)\\ &(f \cdot g)(x_1,\ldots,x_{p+q}) := f(x_1,\ldots,x_p)g(x_{p+1},\ldots,x_{p+q})\end{aligned}$$

We have

$$b(f \cdot g) = (bf) \cdot g + (-1)^p f \cdot (bg)$$

We spell out the definition for $b(f \cdot g)$ in fig. 13. The top row represents $(bf) \cdot g$ and the bottom row represents $f \cdot (bg)$. As a result this multiplication descends to a multiplication in $HH^\bullet(A)$[5].

Two path components in $D_1(2)$ map into two operations in $\mathrm{End}(C^\bullet(A))$, namely, $f \cdot g$ and $g \cdot f$. When embedded into $D_2(2)$, they belong to the same path component. Up to higher homotopy, they are connected by two paths h and h', one

[5] $HH^\bullet(A)$ is a (graded) commutative algebra as a consequence of Gerstenhaber's theorem below.

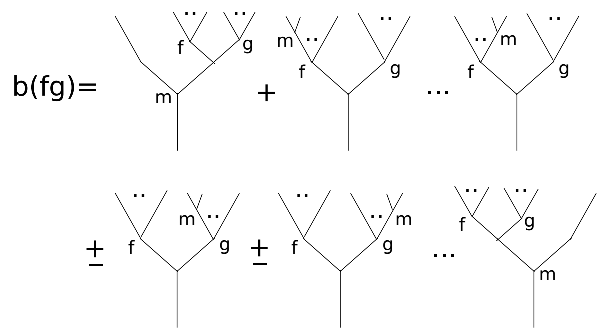


Figure 13: Leibnitz rule in $C^\bullet(A)$.

going clockwise and the other going counter-clockwise. By abuse of notation we denote their image in $\mathrm{Hom}_{\mathrm{dgVect}}(C^\bullet(A)^{\otimes 2}, C^\bullet(A)[-1])$ by the same letters.

Explicitly, given $f \in C^p(A)$ and $g \in C^q(A)$ we define $h_{f,g} \in C^{p+q-1}(A)$ to be

$$h_{f,g}(a_1, \ldots, a_{p+q-1}) := \sum_{1 \le i \le p} (-1)^{(i-1)q} f(a_1, \ldots, g(a_i, \ldots, a_{i+q-1}), \ldots, a_{p+q-1})$$

and $h'_{f,g} := h_{g,f}$. A theorem due to Gerstenhaber states that

$$h_{f,bg} - bh_{f,g} + (-1)^{q-1} h_{bf,g} = (-1)^{q-1} \left(g \cdot f - (-1)^{pq} f \cdot g \right)$$

Bracket: We define a bracket in $C^\bullet(A)$ by

$$[f, g] := h_{f,g} - (-1)^{pq} h_{g,f}$$

descending to a Lie bracket on cohomology.

3.3 Batalin-Vilkovisky Algebras.

Definition: A *Batalin-Vilkovisky algebra* (BV-algebra) is a graded vector space, A, equipped with associative commutative multiplication of degree 0 and a second order differential operator Δ of degree -1, such that the associated bracket is a Lie bracket.

A second order differential operator is an operator such that the associated bracket

$$[x, y] := \Delta(xy) - \Delta(x)y - x\Delta(y)$$

is a derivation in each variable.

A BV-algebra is a G-algebra in which the Lie bracket comes from a second order differential operator and that operator is part of the data.

$H_*(D_1)$	Associative algebra
$C_*(D_1)$	$\mathcal{A}_\infty$ algebra
$H_*(D_1^{\mathrm{fr}})$	Frobenius algebra
$C_*(D_1^{\mathrm{fr}})$	Cyclic $\mathcal{A}_\infty$ algebra
$H_*(D_2)$	Gerstenhaber algebra
$C_*(D_2)$	Homotopy Gerstenhaber algebra
$H_*(D_2^{\mathrm{fr}})$	Batalin-Vilkovisky algebra
$C_*(D_2^{\mathrm{fr}})$	Homotopy Batalin-Vilkovisky algebra

Table 1: Dictionary of Operads and their Algebras.

Example continued: Let X be a smooth manifold and $A = \oplus_i \Gamma(\Lambda^i T_X)$ the algebra of poly-vector fields. As discussed in section 3.1, A is a G-algebra. If X is orientable, and we *choose* a no-where vanishing section of $\Lambda^{\mathrm{top}} T_X^*$, we get an identification $\Gamma(\Lambda^i T_X) \xrightarrow{\sim} \Gamma(\Lambda^{n-i} T_X^*)$. The operator Δ is given by the composition

$$\Gamma(\Lambda^i T_X) \xrightarrow{\sim} \Gamma(\Lambda^{n-i} T_X^*) \xrightarrow{d} \Gamma(\Lambda^{n-i+1} T_X^*) \xrightarrow{\sim} \Gamma(\Lambda^{i-1} T_X)$$

making Δ an enhancement of the Lie bracket.

Framed Little Discs Operad: This operad is the same as the little discs operad only we add a point on the boundary of every boundary circle and require that points match when we compose. We denote this operad by D_2^{fr}. $H_*(D_2^{\mathrm{fr}}(1)) = H_*(S^1) = k[0] \oplus k[1]$. An algebra over $H_*(D_2^{\mathrm{fr}})$ is equipped with a unary operation Δ in degree -1. It is not hard to guess Δ is an enhancement of the Lie bracket.

Claim: An algebra over $H_*(D_2^{\mathrm{fr}})$ is a BV-algebra.

Question: By Deligne's conjecture, $HH^\bullet(A)$ is a G-algebra for every associative algebra A. What restriction should we put on A for $HH^\bullet(A)$ to be a BV-algebra?

For our prototypical example, we needed X to orientable, i.e., we needed a trivialization of $\omega_X = \Lambda^{\mathrm{top}} T_X^*$. In the world of algebraic geometry, a trivialization of ω_X is the Calabi-Yau condition.

Theorem: If A is Frobenius algebra then $HH^\bullet(A)$ is a BV-algebra. Moreover, if A is a cyclic $\mathcal{A}_\infty$ algebra, then $C^\bullet(A)$ is an algebra over $C_*(D_2^{\mathrm{fr}})$.

We will discuss cyclic $\mathcal{A}_\infty$ algebras in lecture 5 in connection with open 2-dimensional topological conformal field theories. A dictionary of operads and their associated algebras appears in table 1.

4 Lecture 4

4.1 Topological Field Theory.

Fix a collection of labels Λ whose elements are referred to as branes.

Definition: We define the category of strings, denoted as Strng. The objects of Strng are finite ordered sets of oriented circles and intervals $I_{A,B}$ with endpoints labeled by elements $A, B \in \Lambda$. The circles are referred to as *closed strings*, and the intervals are referred to as *open strings*.

A morphism between an 'incoming' ordered set and an 'outgoing' one is the following data: an oriented surface with boundary split into two components marked 'in' and 'out' (equipped with induced orientation), an orientation reversing embedding of the 'incoming' set of strings into the 'in' component, and an orientation preserving embedding of the 'outgoing' set of strings into the 'out' component. All this is defined up to diffeomorphism of the surface relative to the boundary.

Those parts of the boundary that are not in the image of the above embeddings are referred to as *free boundaries*. The free boundary components are labeled by elements of Λ.
An example of a morphism in Strng appears in fig. 14. It has two boundary components with embedded incoming closed string and embedded outgoing open string. The outgoing open string starts and end at the same brane (free boundary segment). This imposes the same 'boundary condition' at the endpoints of the open string. One can think of the morphism in fig. 14 as describing the time evolution of a closed string into an open string. At time $t = t'$ the transition from closed to open occurs.

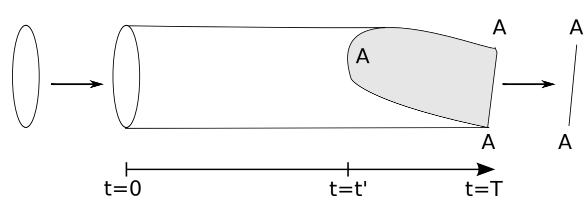
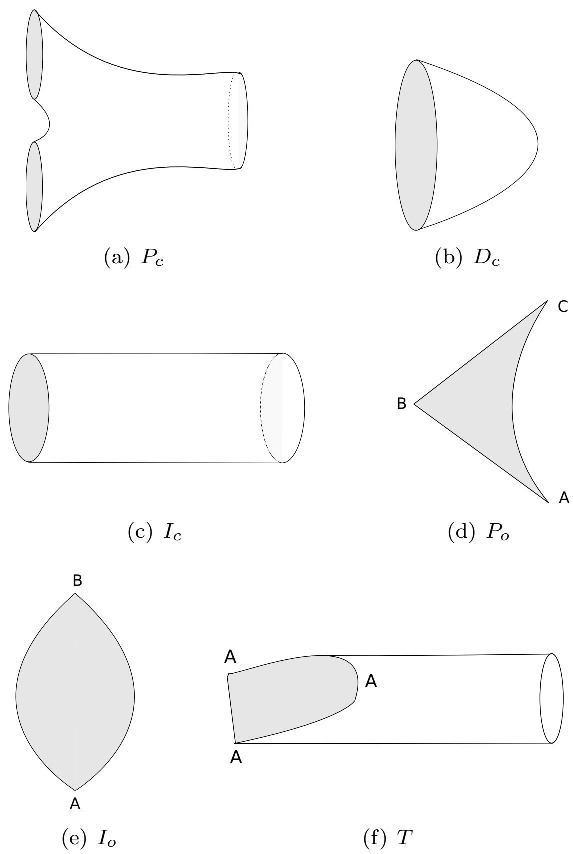


Figure 14: A morphism in Strng

Composition in Strng is given by gluing. Strng is a *symmetric monoidal* category where tensor product is given by disjoint union. It is also *rigid* in the sense that objects have duals, namely, the same manifold with reversed orientation.

Claim: Every morphism in Strng is a composition of morphisms in fig. 15 and their duals.



(a) P_c (b) D_c (c) I_c (d) P_o (e) I_o (f) T

Figure 15: Generators of morphisms in Strng

Fig. 15(a): P_c is a genus 0 surface with 2 incoming and 1 outgoing closed strings.

Fig. 15(b): D_c is a disc with 1 incoming closed string and no outgoing strings.

Fig. 15(c): I_c is a genus 0 surface with 1 incoming and 1 outgoing closed string.

Fig. 15(d): P_o is a disc with 2 incoming and 1 outgoing open strings.

Fig. 15(e): I_o is a disc with 1 incoming and 1 outgoing closed string.

Fig. 15(f): T was discussed in fig 14.

Definition: The category of open strings, $\mathrm{Strng}^{\mathrm{o}}$, is the full subcategory of Strng with objects open strings only. The category of closed strings, $\mathrm{Strng}^{\mathrm{c}}$, is the full subcategory with objects closed strings only.

Definition: A 2-*dimensional open-closed topological field theory* (TFT) is a monoidal functor

$$\Psi : \mathrm{Strng} \to \mathrm{Vect_k}$$

A 2-*dimensional open TFT* is a monoidal functor with $\mathrm{Strng_o}$ as its domain. The same goes for a 2-*dimensional closed TFT*.

Let Ψ be a 2-dimensional open-closed TFT. From now on, when we say TFT we mean open-closed TFT. Let us see what this entails. Denote $V := \Psi(S^1)$. We have $\Psi(P_c) : V \otimes V \to V$ a multiplication on V. We also have $\Psi(D_c) : V \to k$ a trace on V. The composition $\Psi(D_c \circ P_c)$ endows V with a non-degenerate inner product. This makes V into a commutative Frobenius algebra.

Claim: A 2-dimensional closed TFT determines a commutative Frobenius algebra and vice versa.

For $A, B \in \Lambda$ we have $\Psi(I_{A,B}) \in \mathrm{Vect}$. This makes Λ into a k-linear category with

$$\mathrm{Hom}_\Lambda(A, B) := \Psi(I_{A,B})$$

Composition in Λ is given by

$$\Psi(P_o(A, B, C)) : \mathrm{Hom}_\Lambda(A, B) \otimes \mathrm{Hom}_\Lambda(B, C) \to \mathrm{Hom}_\Lambda(A, C)$$

For every $A \in \Lambda$ we have a trace map

$$\mathrm{Hom}_\Lambda(A, A) \xrightarrow{T(A) \sqcup_{S^1} D_c} k$$

and a perfect pairing

$$\mathrm{Hom}_\Lambda(A, B) \otimes \mathrm{Hom}_\Lambda(B, A) \xrightarrow{P_o(A,B,A)} \mathrm{Hom}_\Lambda(A, A) \to k$$

This implies $\mathrm{Hom}_\Lambda(B, A) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(A, B)^\vee$. A k-linear category with such identification is known as a *Calabi-Yau category.*

Claim: A 2-dimensional open TFT determines a Calabi-Yau category and vice versa.

When we combine the above two claims together we get:

Claim: A 2-dimensional TFT determines a pair, (V, Λ), where V is a commutative Frobenius algebra and Λ is a Calabi-Yau category.

For the converse to hold we need some compatibility relations between V and Λ. The whistle diagrams $T(A)$ and $T(A)^\vee$ (see fig. 15) relate open and closed strings. This means we need to have a pair of maps for every $A \in \Lambda$

$$T(A) : \mathrm{End}_\Lambda(A) \to V$$
$$T(A)^\vee : V \to \mathrm{End}_\Lambda(A)$$

The two maps are actually dual when we use the identifications $V \xrightarrow{\sim} V^\vee$ and $\mathrm{End}_\Lambda(A) \xrightarrow{\sim} \mathrm{End}_\Lambda(A)^\vee$ given by the non-degenerate pairings in V and Λ. (We abuse notation when denoting the surface $T(A)$ and its corresponding linear map by the same letters). These maps need to satisfy the following:

- The map $T(A)^\vee : V \to \mathrm{End}_\Lambda(A)$ is a ring homomorphism, and its image is in the center of $\mathrm{End}_\Lambda(A)$.
- The Cardy condition holds. See fig. 16.

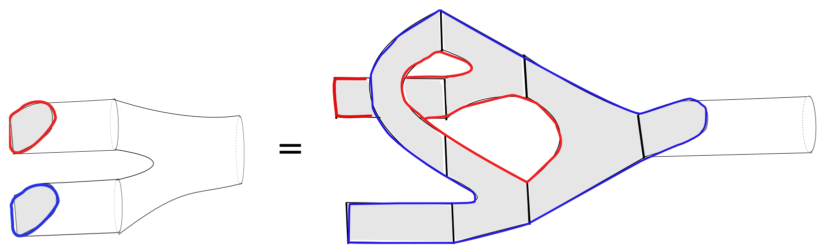


Figure 16: The Cardy condition.

Example: Let G be a finite group. We can construct a 2-dimensional TFT from G with values in $\mathrm{Vect}_\mathbb{C}$. Its associated commutative Frobenius algebra is $V = \mathbb{C}[G]^G$. Elements of V are group algebra elements fixed by the action G on itself by conjugation. These can be thought of as class functions with product given by convolution. The trace map is given by $f \mapsto \frac{1}{|G|} f(e)$. The associated Calabi-Yau category is $\Lambda = \mathrm{Rep}_\mathbb{C} G$. The two are related by

$$\mathbb{C}[G]^G \cong \mathrm{End}(\mathrm{id}_{\mathrm{Rep}G})$$

that is, V is identified with endomorphisms of the identity functor of Λ. A common notation for the endomorphisms of the identity is $Z(\Lambda)$ known as the center of Λ.

The maps arising from $T(A)$ and its dual are given by

$$\begin{array}{cc} \mathrm{End}_{\mathrm{Rep}G}(A) \to \mathbb{C}[G]^G & \qquad \mathbb{C}[G]^G \to \mathrm{End}_{\mathrm{Rep}G}(A) \\ \mathrm{id}_A \mapsto \chi_A & \qquad \eta \to \eta_A \end{array}$$

where $\eta \in \mathbb{C}[G]^G$ is considered as an endomorphism of the identity functor and η_A is its restriction to a representation A.

4.2 Universal TFT.

Question: Every TFT gives rise to an open TFT Ψ^o by restriction to Strng^o. This restriction is known as the *open string sector* of that TFT. The same goes for the *closed string sector* Ψ^c. Given an open TFT, is there an open-closed

TFT with that given open TFT as its open sector?

We have two embeddings

$$i^o : \mathrm{Strng^o} \hookrightarrow \mathrm{Strng}$$
$$i^c : \mathrm{Strng^c} \hookrightarrow \mathrm{Strng}$$

with their associated pullbacks, given by restriction of an open-closed TFT to its open or closed sectors. To answer the above question we would like to construct a left adjoint, $(Li^o)_*$, to $(i^o)^*$. Given Ψ^o an open TFT, $(Li^o)_*(\Psi^o)$ will be the *universal* open-closed TFT with open sector Ψ^o.

Theorem: If Λ is a 2-dimensional open TFT given by a Calabi-Yau category, then there exists a universal open-closed TFT whose open sector is given by Λ and whose closed sector is given by $(i^c)^*(Li^o)_*(\Lambda) := HH^*(\Lambda)$, where $HH^*(\Lambda) := Z(\Lambda) = \mathrm{End}_\Lambda(\mathrm{id}_\Lambda)$.

Symmetric Frobenius Algebra: We consider the above theorem in the simplest case where Λ is a Calabi-Yau category with one object. In this case labellings are redundant. A monoidal category with one object is simply a ring. We will denote it by the same letter $\Lambda := \mathrm{End}_\Lambda(\bullet)$. Λ is a Frobenius algebra, not necessarily commutative. It is endowed with a symmetric pairing coming from the Calabi-Yau structure.

If we want to construct an open-closed TFT compatible with Λ we will need to define a vector space V and a map $T(\bullet) : \Lambda \to V$. In fact, for any diagram with some number of open inputs and a closed output we need to construct such map. The most 'universal' thing we can do is define

$$V = \mathrm{Span}_k \left\{ \begin{matrix} \text{all open-closed diagrams with} \\ \text{open inputs labeled by elements} \\ \lambda \in \Lambda \text{ and one closed output} \end{matrix} \right\} / \sim$$

where we mod out by some relations. Given an open-closed diagram, we can define its corresponding map, going from $\Lambda^{\otimes n}$ to V, by taking an element $\lambda_1 \otimes \cdots \otimes \lambda_n$ to a vector in V given by the same diagram with n incoming open strings labeled λ_1 to λ_n.

The Cardy condition allows us to turn any open-closed diagram into a diagram with an open part glued to a whistle. The Cardy condition needs to be incorporated into V, hence we have

$$V = \mathrm{Span}_k \left\{ \begin{matrix} \text{all open diagrams with open in-} \\ \text{puts labeled by elements } \lambda \in \Lambda \\ \text{\& one whistle} \end{matrix} \right\} / \sim$$

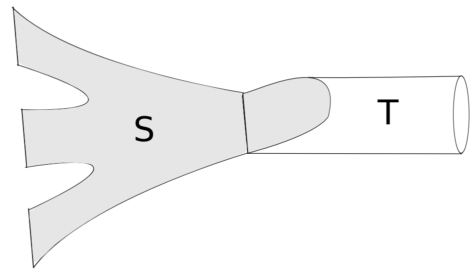


Figure 17: A morphism with open part glued to a whistle.

But now, let us consider an open diagram glued to a whistle, for example, see fig. 17. In any open-closed TFT compatible with Λ, the map associated with $T \circ S$ in has to factor through Λ.

$\lambda_1 \otimes \lambda_2 \otimes \lambda_3$	$\xrightarrow{T \circ S}$	diagram ST with incoming open strings labeled by $\lambda_1, \lambda_2, \lambda_3$
$S \downarrow$		$\|\|$
λ	$\xrightarrow{T}$	whistle diagram with incoming open string labeled by λ

where λ is completely determined by operations in Λ. This implies the two vectors on the right-hand column are equal. Such identities need to be incorporated into V, therefore we have

$$V = \mathrm{Span}_k \left\{ \begin{matrix} \text{all whistle diagrams with open} \\ \text{string labeled by elements } \lambda \in \Lambda \end{matrix} \right\} / \sim$$

Now V carries a multiplication induced from Λ. But the multiplication in V has to be commutative. We should have a surjection

$$\Lambda/[\Lambda, \Lambda] \to V$$

We argue that the universal TFT, with Λ its open sector, is given by the pair $(\Lambda, \Lambda/[\Lambda, \Lambda])$.

Note that the trace defined on $\Lambda/[\Lambda, \Lambda]$ is not induced from Λ. The trace in V is defined by the diagram in fig. 18. Let us compute it explicitly. Given the perfect pairing in Λ we can choose an orthonormal basis $\{e_i\}$. We can write

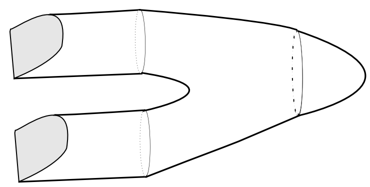


Figure 18: Trace in V

down the map associated to fig. 18 using the Cardy relation.

$$\begin{aligned}\lambda \otimes \lambda' &\mapsto \sum_i \lambda \otimes e_i \otimes \lambda' \otimes e_i \\ &\mapsto \sum_i \lambda e_i \otimes \lambda' e_i \\ &\mapsto \sum_i \mathrm{tr}_\Lambda(\lambda e_i \lambda' e_i) \\ &= \sum_i \langle \lambda e_i \lambda', e_i \rangle_\Lambda\end{aligned}$$

This is exactly the trace of the map $x \mapsto \lambda x \lambda'$.

Conclusion: Let Λ be a symmetric Frobenius algebra and assume $\langle \lambda, \lambda' \rangle := \mathrm{Tr}(x \mapsto \lambda x \lambda')$ is a non-degenerate pairing on $V := \Lambda/[\Lambda, \Lambda]$. Then (Λ, V) is the universal open-closed TFT with open sector Λ.

Note that $HH_0(\Lambda) = \Lambda/[\Lambda, \Lambda]$. The Calabi-Yau condition allows us to identify Hochschild homology with co-homology. In particular, $Z(\Lambda) \cong \Lambda/[\Lambda, \Lambda]$. This agrees with the theorem we previously stated.

For a general Calabi-Yau category we $HH_i(\mathcal{C}) \cong HH^{n-i}(\mathcal{C})$. In fact, the Calabi-Yau condition gives an isomorphism on the level of chain complexes.

5 Lecture 5

5.1 Moduli Spaces of Curves.

Let $\mathcal{M}_{g,n}$ denote the moduli space of genus g stable curves with n marked points[6]. We should think of stable curves as surfaces that admit a metric with constant curvature -1, i.e., hyperbolic. We have the following

$$\begin{aligned} \mathcal{M}_{0,3} &= \{\text{pt}\} \\ \mathcal{M}_{0,4} &= \mathbb{P}^1 \backslash \{3 \text{ pts}\} \\ \mathcal{M}_{0,5} &= \mathbb{P}^1 \times \mathbb{P}^1 \backslash \{7 \text{ lines}\} \\ \mathcal{M}_{1,1} &= \mathbb{A}^1 \quad \text{(the } j\text{-line)} \end{aligned}$$

The Deline-Mumford compactification of the above is

$$\begin{aligned} \overline{\mathcal{M}_{o,3}} &= \{\text{pt}\} \\ \overline{\mathcal{M}_{0,4}} &= \mathbb{P}^1 \\ \overline{\mathcal{M}_{0,5}} &= \mathbb{P}^2 \text{ blown up at 4 points} \\ \overline{\mathcal{M}_{1,1}} &= \mathbb{P}^1 \end{aligned}$$

The Deligne-Mumford compactification is constructed by adding degenerations. For example, when two punctures on a sphere collide we get a bubbling phenomena as in fig. 19. In the case of $\mathcal{M}_{1,1}$ we add a nodal curve given by $x^2 = y^2(y-1)$.

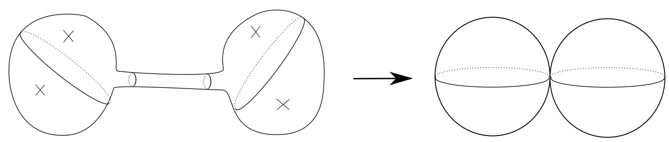


Figure 19: Bubbling.

5.2 Ribbon Graphs.

Definition: A *Ribbon Graph* (RG), Γ, is a graph with a cyclic ordering of half edges around each vertex. We assume the valency at each vertex is at least 3. A few examples are given in figure 20.

[6]In what follows marked points will sometimes be labeled and sometimes not, this distinction will be important when we discuss operads.

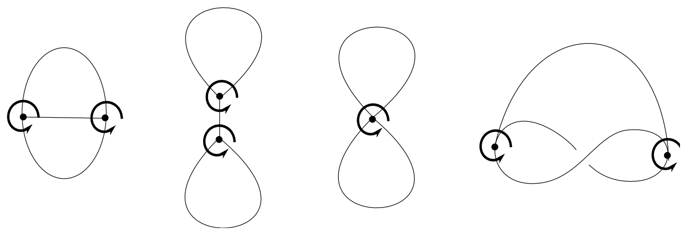


Figure 20: Ribbon Graphs.

Note that a ribbon graph is the same data as two permutations σ and τ in $\Sigma_{2\times\text{edges}}$ such that $\sigma^2 = \text{id}$ and has no fixed points. σ encodes how half edges are associated. τ encodes the cyclic data at the vertices. For example, consider a ribbon graph as in fig. 21.

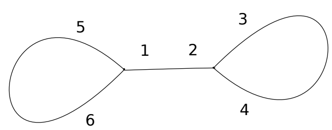


Figure 21: Ribbon Graph Permutations.

The permutations associated to it are

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix} \quad , \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 2 & 3 & 6 & 1 \end{pmatrix}$$

We can identify

$$\begin{array}{ccccc} \text{Vertices } V \text{ of } \Gamma & = & \text{cycles of } \tau & = & (1,5,6)(2,4,3) \\ \text{Edges } E \text{ of } \Gamma & = & \text{cycles of } \sigma & = & (1,2)(3,4)(5,6) \\ \text{Faces } F \text{ of } \Gamma & = & \text{cycles of } \tau\sigma & = & (1,6,2,3)(4)(5) \end{array}$$

We define the *genus* of Γ to be the number g such that $2-2g = V - E + F$. The genus of the first three ribbon graphs in fig. 20 is zero, the genus of the last one is 1. Note g is the smallest genus surface on which Γ can be drawn without intersecting edges. We also define the *degree* of Γ to be $\deg(\Gamma) = \sum_{v\in V}(\text{valence}(v) - 3)$. The degree of the first two ribbon graphs in fig. 20 is zero, the degree of the third is 1.

RG Complex: We construct a complex with underlying complex vector space given by the span of all ribbon graphs modulo the appropriate notion of isomorphism. This vector space can be graded by the degree. We denote this graded vector space by RG and define a map

$$d : RG_n \to RG_{n-1}$$
$$d\Gamma = \sum_{\Gamma'/e=\Gamma} \pm\Gamma'$$

where we sum over all pairs $(\Gamma', e \in E(\Gamma'))$ such that collapsing e in Γ' to a point produces Γ. We claim that, given the right choice of signs, this is a differential. We denote this complex by (RG, d) and refer to it as the *ribbon graph complex*. The differential we just defined does not change the genus of Γ or the number of faces (contracting an edge is a local operation). Hence we can fix g and F and consider $(RG(g, F), d)$. The significance of ribbon graphs is apparent in the following theorem.

Theorem: [Strebbel & Penner] The ribbon graphs complex $(RG(g, n), d)$ is quasi isomorphic to $C_*(\mathcal{M}_{g,n})$ for $n \geq 1$.

Let us consider the case $g = 0, F = 3$. We have $V - E = -1$. Comparing edges to vertices we have $3V \leq 2E$. This implies $V \leq 2$[7]. We find two ribbon graphs in degree zero, and one ribbon graph in degree 1. The complex is given by $\mathbb{C} \to \mathbb{C}^2 \to 0$. The differential is shown in fig. 22. It is clear $H_*(\mathcal{M}_{0,3}) \cong H_*(\mathrm{RG}(0, 3), d)$, though perhaps not a very convincing demonstration of the theorem.

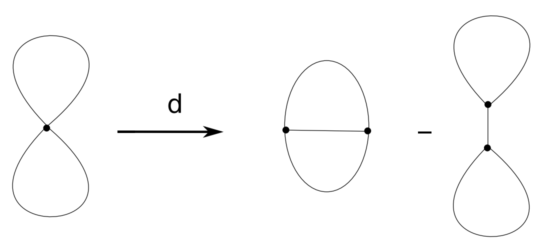


Figure 22: Differential for $RG(0, 3)$.

Metric RG: A *metric ribbon graph* is a ribbon graph with an assignment of positive length to each of its edges. We denote by $\mathcal{M}_{g,n}^{\mathrm{comb}}$ the moduli space of metric ribbon graphs. We topologize it by gluing pieces of $\mathbb{R}_+^{\#E(\Gamma)}$ assigned to each $\Gamma \in \mathcal{M}_{g,n}$ via contraction of edges.

For every face $f \in F(\Gamma)$ of a metric ribbon graph Γ we define its diameter

$$\mathrm{diam(f)} := \sum_{\mathrm{e}\in\partial\mathrm{f}} \mathrm{length(e)}$$

[7] A similar computation shows $RG(g, F)$ is finite dimensional at each degree for all g, F.

We have a homeomorphism $\mathcal{M}_{0,3}^{\text{comb}} \xrightarrow{\sim} \mathbb{R}_+^3$ given by the diameter of each face.

Theorem: [Mumford, Penner, Strebbel] We have a homeomorphism

$$\mathcal{M}_{g,n}^{\text{comb}} \cong \mathcal{M}_{g,n} \times \mathbb{R}_+^n$$

5.3 Topological Conformal Field Theory.

Definition: Fix a collection of labels Λ whose elements are referred to as branes. Let Strng^{cx} denote the category with objects the same as Strng and morphisms are equipped with conformal structure (for the definition of Strng see section 4.1). The category $C_*(\text{Strng}^{\text{cx}})$ has the same objects as Strng and morphisms are given by

$$C_*(\text{Strng}^{\text{cx}})(\text{X}, \text{Y}) := \text{C}_*(\text{Strng}^{\text{cx}}(\text{X}, \text{Y}))$$

Note that $\text{Strng} = \text{H}_0(\text{C}_*(\text{Strng}^{\text{cx}}))$, in other words, the path components of $\text{Strng}^{\text{cx}}(\text{X}, \text{Y})$ are determined by the topological type.

Definition: A *conformal field theory* (CFT) is a monoidal functor from Strng^{cx} to some monoidal target category.

We have no example of such a theory to date. Instead we consider theories which are half way between CFT and TFT.

Definition: We define a 2-dimensional *topological conformal field theory* (TCFT) to be a monoidal functor of dg-categories $C_*(\text{Strng}^{\text{cx}}) \to \text{dgVect}$.

Given $A, B \in \Lambda$, a TCFT associates to each object $I_{A,B}$ a differential graded vector space which we denote by

$$\text{Hom}_\Lambda^\bullet(A, B) := \Psi(I_{A,B})$$

As in the case of the open-closed TFT we consider P_o in fig. 15(d) for our notion of composition in Λ (see section 4.1). Let $\mathcal{M}_{P_o}$ denote the moduli space of Riemann surfaces of topological type P_o. Applying Ψ we get a map of dg-vector spaces

$$C_*(\mathcal{M}_{P_o}) \otimes \text{Hom}^\bullet(A, B) \otimes \text{Hom}^\bullet(B, C) \to \text{Hom}^\bullet(A, C)$$

We no longer have a composition of morphisms in Λ but rather a differential graded space worth of them.

We will dedicate the rest of this lecture to a discussion of a theorem due to Costello which is a spruced up version of a theorem by the same author we encountered earlier on in section 4.2. Replacing TFT by TCFT requires passing to the $\mathcal{A}_\infty$ world. The adjoint functor to $(i^o)^*$ should now be derived and we need a notion of Hochschild chain complex for this setting.

Theorem: [Costello] A 2-dimensional open TCFT determines a cyclic $\mathcal{A}_\infty$ category Λ and vice versa. Given a cyclic $\mathcal{A}_\infty$ category Λ there exists a universal open-closed TFT whose open sector is given by Λ and whose closed string sector is given by $C^*(\Lambda)$, the Hochschild cochain complex of Λ.

Cyclic $\mathcal{A}_\infty$ Algebra: We consider the above theorem in the simplest case where Λ is a cyclic $\mathcal{A}_\infty$ category with one object. In this case the endomorphisms of that single object form a cyclic $\mathcal{A}_\infty$ algebra.

A *cyclic $\mathcal{A}_\infty$ algebra* is an $\mathcal{A}_\infty$ algebra, A, with a non-degenerate pairing $\langle , \rangle : A \otimes A \to k$ such that the following maps

$$\begin{aligned} &c_{n+1} : A^{\otimes(n+1)} \to k \\ &x_1 \otimes \cdots \otimes x_{n+1} \mapsto \langle m_n(x_1, \ldots, x_n), x_{(n+1)} \rangle \end{aligned}$$

are invariant under cyclic permutation (up to signs depending on degrees). For example, a cyclic $\mathcal{A}_\infty$ algebra with only m_2 is a symmetric Frobenius algebra.

Invariants of Ribbon Graphs: A cyclic $\mathcal{A}_\infty$ algebra, A, gives rise to a numerical invariant, $\mathrm{ev}(\Gamma, \mathrm{A})$, associated to every ribbon graph Γ. We will demonstrate the construction with the following example. Consider a ribbon graph Γ with a morsification as in fig. 23.

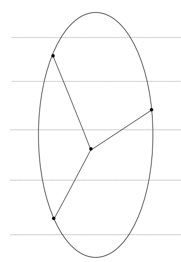


Figure 23: Morsification of a ribbon graph Γ.

We define $\mathrm{ev}(\Gamma, \mathrm{A})$ to be the composition of the following maps

$$\begin{aligned} \mathbb{C} \xrightarrow{\langle , \rangle^*} A \otimes A \xrightarrow{\mu^* \otimes \mathrm{id}} A \otimes A \otimes A \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes \mu^*} A \otimes A \otimes A \otimes A \\ \xrightarrow{\mathrm{id} \otimes \mu \otimes \mathrm{id}} A \otimes A \otimes A \xrightarrow{\mu \otimes \mathrm{id}} A \otimes A \xrightarrow{\langle , \rangle} \mathbb{C} \end{aligned}$$

We claim $\mathrm{ev}(\Gamma, \mathrm{A})$ does not depend on the choice of morsification. Moreover, this calculation can be generalized to ribbon graphs with vertices of valency greater than 3. In that case we use $c_k : A^k \to A$ to get a map $A^{\otimes i} \to A^{\otimes(k-i+1)}$. A has a non-degenerate pairing hence a finite basis $\{e_i\}$. The above definition agrees

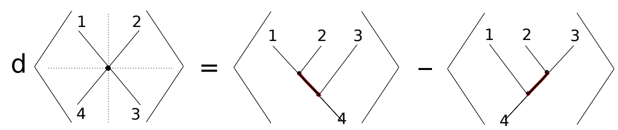


Figure 24: Vertex Expansion.

with the following.

$$\mathrm{ev}(\Gamma, \mathrm{A}) = \sum_{\substack{\text{labeling of half edges} \\ \text{of } \Gamma \text{ by elements of basis}}} \prod_{\substack{\text{edges with halves} \\ \text{labeled by } e_i, e_j}} \pm\langle \mathrm{e_i}, \mathrm{e_j} \rangle \prod_{\substack{\text{vertices with adjacent 1/2-edges} \\ \text{labeled } e_{i_1}, \cdots, e_{i_k} \text{ in this cyclic order}}} \mathrm{c_k}(\mathrm{e_{i_1}}, \cdots, \mathrm{e_{i_k}})$$

We ignored the issue of signs in this expression. We claim this expression does not depend on choice of basis.

Theorem: $\mathrm{ev}(\mathrm{d}\Gamma, \mathrm{A}) = 0$. This means every cyclic $\mathcal{A}_\infty$ algebra defines a cocycle on (RG, d). In particular, it defines a class in $H^*(\mathcal{M}_{g,n})$.

Taking the differential at a vertex of Γ of valence $n+1$ we get all trees with n leaves, one root, and one internal edge. For example, when Γ contains a vertex of valence 4, see fig. 24. The bracket in the picture is meant to remind us that we are looking at a part of Γ containing that vertex. Evaluation at an expanded vertex corresponds exactly to Assoc_n and hence equals zero.