

# Mirror Symmetry with D-branes

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## Lecture One

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### Calabi-Yau manifold

There's a confusion of the definition of Calabi-Yau manifold among the mathematicians and physicists. Here're the definitions from both side:

**Definition** (For Mathematician). *A Calabi-Yau 3-fold  $X$  is a compact Kahler 3-dimensional manifold with Ricci flat Kahler metric.*

**Definition** (For Physicist). *A Calabi-yau 3-fold  $X$  is a compact 3-dimensional complex manifold with Kahler metric such that the holonomy group  $G \subset SU(3)$  but not contained in any  $SU(2)$  subgroup of  $SU(3)$ .*

**Remark.**  $G \not\subset SU(2)$  is a really serious condition for physics since otherwise it would change the supersymmetry.

The following theorem of Yau allows us to easily construct many examples of Calabi-Yau manifolds.

**Theorem** (Yau). *If  $X$  is compact Kahler with trivial canonical line bundle. Then given any Kahler metric  $g$  on  $X$ , there exists an unique Kahler metric  $\tilde{g}$  which has the same Kahler class as  $g$  such that  $\tilde{g}$  is Ricci flat.*

### Examples of Calabi-Yau manifolds

- (1)  $X = \mathbb{C}^n / \Lambda$  torus.  $\Lambda \simeq \mathbb{Z}^{2n}$  is a discrete subgroup of  $\mathbb{C}^n$ .
- (2) Complete intersections in Fano varieties. Let  $Y$  be compact Kahler,  $s \in H^0(Y, K_Y^{-1}) \neq 0$ . Suppose

$$X_s = \{s = 0\}$$

is smooth. Then by Adjunction formula,  $K_X$  is trivial. Hence by Yau's theorem, it admits a Ricci flat metric. More specifically, we have

- (a) Cubic curve in  $\mathbb{P}^2$ .
- (b)  $X = \{s = 0\}$ ,  $s \in H^0(\mathbb{P}^3, O(4))$ . This is a K3 surface
- (c)  $T = \{f_1 = f_2 = f_3 = 0\}$ ,  $f_i \in H^0(\mathbb{P}^5, O(2))$ .  $T$  is another example of K3.

- (3) **Enrique Surface:** Not-simply-connected compact complex surface with universal cover a K3 surface. A theorem of Kodaira says that if  $X$  is Enrique surface, then  $\pi_1(X) = \mathbb{Z}/2\mathbb{Z}$ . Here's an explicit example: Take  $\mathbb{P}^5$  with homogeneous coordinate  $x_1, x_2, x_3, x_4, x_5, x_6$ , and choose  $F_1, F_2, F_3$  homogeneous of degree 2 in  $x_1, x_2, x_3$ , and  $G_1, G_2, G_3$  homogeneous of degree 2 in  $x_4, x_5, x_6$ . Let

$$X = \{F_1 - G_1 = F_2 - G_2 = F_3 - G_3 = 0\}$$

There's an evolution

$$\sigma : \mathbb{P}^5 \rightarrow \mathbb{P}^5, \quad [x_1, x_2, x_3, x_4, x_5, x_6] \rightarrow [x_1, x_2, x_3, -x_4, -x_5, -x_6]$$

$X$  is preserved by  $\sigma$  and  $X/\sigma$  gives an example of Enrique surface. Note that the Ricci flat metric on  $X$  descends to give a Ricci flat metric on  $X/\sigma$ , but  $K_{X/\sigma}$  is torsion and not trivial.

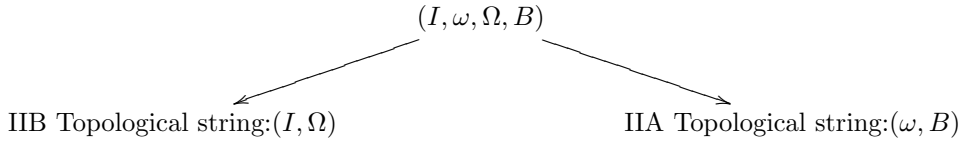
## Statement of Mirror Symmetry

Roughly speaking, Mirror Symmetry identifies two string compactifications near a special boundary point in the moduli space of CY 3-fold.

Let  $X$  be CY 3-fold. It has geometric structure

$$X = (M, I, \omega, \Omega, B)$$

where  $M$  is the underlying smooth structure as manifold, and  $I$  is the complex structure,  $\omega$  is Ricci-flat kahler form,  $\Omega$  is no-where vanishing holomorphic 3-form,  $B \in H^2(M, S^1)$  is called B-field. String compactification of Type II theory on  $M$  gives two topological theories with different field contents:



**Rough Mirror Symmetry:** If  $M, \check{M}$  are two underlying manifolds of CY 3-folds. Consider the moduli space

$$\begin{aligned}
 M_B &= \text{date } (I, \Omega) \text{ on } M \\
 \check{M}_A &= \text{date } (\check{\omega}, B) \text{ on } M
 \end{aligned}$$

Assume that there exist special boundary points on the moduli space (to be explained later)

$$\begin{aligned}
 \text{large volume limit} & : \infty_A \in \check{M}_A \\
 \text{large complex structure limit} & : \infty_B \in M_B
 \end{aligned}$$

We say that  $M, \check{M}$  are in **Mirror Symmetry** if there're neighborhoods  $\check{U}_A \ni \infty_A, U_B \ni \infty_B$  and an isomorphism

$$\varphi : \check{U}_A \rightarrow U_B$$

such that we have isomorphism of TQFT (Topological Quantum Field Theory)

$$IIA(\check{\omega}, \check{B}) \simeq IIB(\varphi(\check{\omega}, \check{B}))$$

$\varphi$  is called **mirror map**.

## Large Complex Structure Limit

Suppose  $(X, \Omega)$  is algebraic complex compact  $n$ -fold with no-where vanishing holomorphic volume form.

**Definition.** A *degeneration* of  $(X, \Omega)$  is a pair  $(\mathcal{X} \xrightarrow{f} \Delta, \eta)$  where  $\Delta = \{q \in \mathbb{C} \mid |q| < r\}$  is complex 1-dim disk,  $\mathcal{X}$  is smooth complex manifold of dimensional  $n + 1$ ,  $f$  is holomorphic proper map,  $\eta \in H^0(\mathcal{X}, \Omega_f^n)$  no-where vanishing section of the relative differential  $n$ -form, such that

- (a)  $f$  is submersion over  $\Delta^x = \Delta - \{0\}$
- (b)  $\exists$  base point  $b \in \Delta^x$  such that  $(\mathcal{X}_b, \eta_b) \simeq (X, \Omega)$ . Here we use the notation  $\mathcal{X}_b = f^{-1}(b)$ .

**Theorem** (Morrison, Kontsevich, Gross-Wilson).

- (1)  $\exists c \in \mathbb{C}, c \neq 0, k \in \mathbb{Z}, 0 \leq m \leq n$ , such that

$$\int_{\mathcal{X}_b} \eta_q \wedge \bar{\eta}_q = c(\log |q|)^m |q|^{2k} (1 + O(1)) \quad q \rightarrow 0$$

- (2)  $m = n \Leftrightarrow T : H^n(X, \mathbb{Z}) \rightarrow H^n(X, \mathbb{Z})$  is maximal quasi-unipotent, i.e.,

$$\exists s \text{ s.t. } (T^s - id)^{n+1} = 0, (T^s - id)^n \neq 0$$

Here  $T$  is the monodromy operator.

**Remark.** By a theorem of Borel-Deligne, for any degeneration  $\mathcal{X} \rightarrow \Delta$  of compact Kahler manifolds, the monodromy  $T$  is quasi-unipotent such that  $(T^s - id)^d = 0$  for some  $s$  and some  $d \leq n + 1$ . The condition that  $m = n$  justifies the terminology "maximal".

**Definition.**  $\mathcal{X} \rightarrow \Delta$  is a large complex structure limit iff it's maximal unipotent degeneration (i.e.  $m = n$ ).

**Remark.** If  $X \subset \mathbb{P}^n$  is complete intersection CY, then there exists large complex structure degeneration of  $(X, \Omega)$  and the limit  $\mathcal{X}_0$  is unique. The uniqueness of this limit may not be true if  $X$  is in general complete intersection in toric orbifold.

## Homological Mirror Symmetry Conjecture

Now we want to work with open string theories and add D-branes to our mirror story. We will focus on the category of topological D-brane. Let  $(M, I, \Omega, \omega, B)$  be the data of CY 3-fold. We can associate two categories **Topo A** and **Topo B**, where

**Topo A** : Boundary Topological field theory compactible with IIA-theory

**Topo B** : Boundary Topological field theory compactible with IIB-theory

Note that **Topo A** is roughly the category of A-branes, which depends only on  $\omega + \sqrt{-1}B \in H^2(X, \mathbb{C}^*)$ , while **Topo B** is the category of B-branes, which depend only on  $(I, \Omega)$ . We want **Topo A** and **Topo B** to be some moduli stacks (or derived stacks). In fact there're categories of topological branes, which we call A-model and B-model respectively

**A – model** :  $DFuk(M, \omega + \sqrt{-1}B)$  Derived Fukaya Category

**B – model** :  $D^bCoh(M, I)$  Derived category of coherent sheaves

Note that the dependence of B-model on the homomorphic top form is encoded in the serre functor on  $D^bCoh(M, I)$ .

## Homological MS Conjecture

Suppose we have A and B model data

B-model :  $(X, \Omega)$  – compact complex CY 3-fold with a holomorphic volume form

A-model :  $(Y, \omega)$  – real compact 6-dim symplectic manifold which underlies a Ricci flat which underlies a Ricci flat Kahler metric on CY 3-fold

We say that  $(X, \Omega)|(Y, \omega)$  are in HMS with each other if there exists  $\Delta = \{q \in \mathbb{C} | |q| < r\}$ ,  $b \in \Delta^x$  and large complex structure degeneration  $(\mathcal{X} \rightarrow \Delta, \eta)$  with  $(\mathcal{X}_b, \eta_b) \simeq (X, \Omega)$  such that for any  $q \in \Delta^x$  we have an equivalence of categories

$$HMS_q : D^b Coh(\mathcal{X}_q) \xrightarrow{\simeq} DFuk(Y, q\omega)$$

Note that the large volume limit on A-model here is simply the trivial limit  $q\omega, q \rightarrow 0$ .

## Two Refinements of HMS

### Refinement 1–Categorical

Categorically,  $D^b Coh(\mathcal{X}_q)$  and  $DFuk(Y, q\omega)$  are moded-out versions of the categories of branes. More precisely, the actual category of A-branes is the Fukaya category

$$Fuk(Y, q\omega)$$

which is  $A_\infty$ -category.  $DFuk$  is the homotopy category of  $Fuk$ . On the other side, the actual category of B-branes is

$$\mathcal{P}(X) = \{\text{flat } (0, \cdot) \text{ superconnections on } X\}$$

Here the object of  $\mathcal{P}(X)$  is given by

$$\{E^\bullet = \bigoplus_{i=a}^b E^i, \nabla\}$$

where  $E^i$  is smooth complex vector bundles,

$$\nabla : E^\bullet \otimes \mathcal{A}_X^{(0, \bullet)} \rightarrow E^\bullet \otimes \mathcal{A}_X^{(0, \bullet)}$$

is  $\mathbb{C}$ -linear of total degree=1, where  $\mathcal{A}_X^{(0, \bullet)}$  is the complex of anti-homomorphic differential forms. It satisfies the Leibniz rule

$$\nabla(e a) = \nabla(e) a + (-1)^{\deg e} e \otimes \bar{\partial} a, \quad a \in \mathcal{A}_X^{(0, \bullet)}, e \in E^\bullet$$

the flatness of  $\nabla$  simply means that

$$\nabla^2 = 0$$

Given two objects  $(E, \nabla), (F, \delta) \in \mathcal{P}(X)$ , we can form the Hom complex

$$\underline{\text{Hom}}_0(E^\bullet, F^\bullet) \otimes \mathcal{A}_X^{(0, \bullet)}$$

with induced tensor product connection as usual.

The category  $\mathcal{P}(X)$  is a natural generalization of the following structure: given  $(E^\bullet, f^\bullet)$  a finite complex of holomorphic vector bundles

$$E^1 \xrightarrow{f^1} E^2 \xrightarrow{f^2} E^3 \dots \xrightarrow{f^{k-1}} E^k$$

and consider the following connection

$$\nabla = \begin{pmatrix} \bar{\partial}_{E^1} & & & & \\ f^1 & \bar{\partial}_{E^2} & & & \\ & \cdots & \cdots & & \\ & & f^{k-1} & \bar{\partial}_{E^n} & \end{pmatrix}$$

then  $\nabla^2 = 0$  is equivalent to

$$\bar{\partial}_{E^i}^2 = 0, \quad f^{i+1}\bar{\partial}_{E^i} = \bar{\partial}_{E^{i+1}}f^i, \quad f^{i+1} \circ f^i = 0$$

Now if we take degree=0 part of Hom as morphisms of objects in  $\mathcal{P}(X)$ , we can form a homotopy category  $H_0(\mathcal{P}(X))$ , and there's a natural functor

$$D^b Coh(X) \rightarrow H_0(\mathcal{P}(X))$$

**Theorem** (J.Block). *This functor is an equivalence if  $X$  is projective.*

With the same notation as before, the refined version of HMS says that

$$HMS_q : \mathcal{P}(\mathcal{X}_q) \xrightarrow{\cong} Fuk(Y, q\omega)$$

is quasi-equivalence of  $A_\infty$ -categories. It implies the original HMS after taking the homotopy category.

### Refinement 2–Geometric

In the set-up of refinement 1, the moduli space is formulated using  $A_\infty$ -categories. In physics, however, we have two bigger moduli spaces, which we call

$$\begin{aligned} \text{Branes}_A^{off-shell} & : \text{off-shell deformations of } (L, \delta) \\ \text{Branes}_B^{off-shell} & : \text{off-shell deformations of } (E^\bullet, \nabla) \end{aligned}$$

where  $\text{Branes}_A^{off-shell}$  and  $\text{Branes}_B^{off-shell}$  are complex manifolds. We have two superpotentials, both are holomorphic functions

$$\begin{aligned} \omega_A & : \text{Branes}_A^{off-shell} \rightarrow \mathbb{C} \\ \omega_B & : \text{Branes}_B^{off-shell} \rightarrow \mathbb{C} \end{aligned}$$

Then refinement 2 of HMS can be stated as:

**Conjecture:** If  $(X, \Omega)|(Y, \omega)$  are in HMS, with mirror map

$$(E^\bullet, \nabla) \xleftrightarrow{HMS_q} (L, \delta)$$

then there exists moduli space  $\text{Branes}_A^{off-shell}$ ,  $\text{Branes}_B^{off-shell}$  with super-potential  $\omega_A, \omega_B$  such that

$$\begin{aligned} \text{Topo A} = (d\omega_A = 0) & : \text{moduli space of deformations in } Fuk(Y, q\omega) \\ \text{Topo B} = (d\omega_B = 0) & : \text{moduli space of deformations in } \mathcal{P}(\mathcal{X}_q) \end{aligned}$$

Moreover,

- (1)  $HMS_q : \text{Topo A} \rightarrow \text{Topo B}$  is an isomorphism
- (2)  $\omega_A$  is constant on connected components of Topo A,  $\omega_B$  is a constant on connected components of Topo B
- (3)

$$\omega_A|_{\text{Topo A}} = \omega_B|_{\text{Topo B}} + \sum_{i=1}^3 c_i (\log q)^i \quad c_i \in \mathbb{Z}$$

as functions of  $q$ .

## Off-Shell Moduli Space And Superpotential

### B-model

B-model side is easier to describe. In fact,

$$\text{Branes}_B^{\text{off-shell}} = \{\text{all superconnections on } E^\bullet\} = \Gamma_{L^2}(X, \left( \text{End}(E^\bullet) \otimes \mathcal{A}_X^{0,\bullet} \right)^1)$$

where the last isomorphism is non-canonical, and we have used the  $L^2$ -complete sections. The superpotential is given by holomorphic Chern-Simons functional

$$\omega_B : A \rightarrow \int_X \text{Tr} \left( \bar{\partial}A \wedge A + \frac{2}{3}A^3 \right) \wedge \Omega$$

it's easy to check that

$$\frac{\delta \omega_B}{\delta A} = 0 \leftrightarrow (\bar{\partial} + A)^2 = 0$$

### A-model

The A-model side is tricky. Take a classical object  $(L, \delta^{cl})$  in  $\lim_{q \rightarrow 0} Fuk(Y, q\omega)$ , where

$$\begin{aligned} L \subset Y & \quad \text{-- Lagrangian for } \omega \\ \delta^{cl} & \quad \text{-- flat (super)connection on complex of rank n vector bundles on } L \end{aligned}$$

We would like to define "off-shell" deformations of  $(L, \delta^{cl})$  as quantum deformation corresponding to instanton corrections. The first observation is that since  $\delta^{cl}$  is flat, we can view  $(V, \delta^{cl})$  as a dg-module over the dg algebra of differential forms  $(\mathcal{A}_L^\bullet, d, \wedge)$  on  $L$ , where the module is given by

$$(V \otimes \mathcal{A}_L^\bullet, d^{\delta^{cl}})$$

Then the idea is that we can deform  $(\mathcal{A}_L^\bullet, d, \wedge)$  by instanton corrections and we check that there exists a unique compatible deformation of  $(V, \delta^{cl})$  as  $A_\infty$ -modules, which gives  $(V, \delta)$  which can be viewed as (super)connections on  $L$ . The drawback is that we know little about its detailed information except for the existence. More precisely

We describe Quantum corrections: Let

$$\begin{aligned} \mathcal{M}_{n+1,L} & = \text{moduli space of pseudo-holomorphic disks on } (Y, L) \\ & \quad \text{with } n+1 \text{ marked points on the boundary} \end{aligned}$$

$(f, D) \in \mathcal{M}_{n+1,L}$  is given by a continuous map from holomorphic disk

$$f : D \rightarrow Y$$

such that

$$f|_{D-\partial D} \text{ is pseudo-holomorphic, } f(\partial D) \subset L, x_1, \dots, x_m \in \partial D$$

### Remark.

- (1) disk can bubble off, therefore we have to look at suitably defined stable maps
- (2) In the case  $\dim_{\mathbb{R}} Y = 6$ ,  $\text{Vir} \dim \mathcal{M}_{n+1,L} = n+1$ , and  $\mathcal{M}_{n+1,L}$  is a smooth DM stack which has virtual fundamental class
- (3)  $\mathcal{M}_{n+1,L}$  is disconnected,  $\pi_0(\mathcal{M}_{n+1,L}) = \pi_0(\mathcal{M}_{0,L})$

We have natural evaluation map

$$ev : \mathcal{M}_{n+1,L} \rightarrow L^{n+1}, (f, \{x_i\}) \rightarrow (f(x_1), \dots, f(x_{n+1}))$$

which defines a map

$$\begin{aligned} \mathcal{A}_L^{\bullet \otimes n+1} &\rightarrow \mathbb{C} \\ \alpha &\rightarrow \int_{ev_*[\mathcal{M}_{n+1,L}]^{vir}} \alpha \end{aligned}$$

which can be equivalently described as a map

$$\varphi : \mathcal{A}_L^{\bullet \otimes n} \rightarrow \mathcal{A}_L^{\bullet}$$

This defines an  $A_\infty$ -deformation of  $(\mathcal{A}_L^{\bullet}, d, \wedge)$  to  $(\mathcal{A}_L^{\bullet}, \{m_n^q\})$ , where

$$m_n^q = m_n^{cl} + \sum_{\eta \in \pi_0(\mathcal{M}_{n+1,L})} \varphi_\eta q^{a_\eta}, \quad a_\eta = \int_D f^* \omega$$

note that

$$m_n^{cl} = \begin{cases} d & n = 1 \\ \wedge & n = 2 \\ 0 & n \geq 3 \end{cases}$$

**Theorem** (Fukaya-Oh). *There exists an unique deformation  $(V, \delta)$  of  $(V, \delta^{cl})$  as a module over  $(\mathcal{A}_L^{\bullet}, \{m_n^q\})$*

Now we can define

$$\text{Brane}_A^{off-shell} = \text{all } q\text{-deformations of } (L, \delta^{cl}) \text{ as (super)connection on } L$$

we can define the quantum chern-simons action

$$QCS(A, q) = \int_L \sum_{n \geq 0} \text{Tr}(m_n^q(A, \dots, A), A)$$

then the A-model superpotential is given by

$$\omega_A(A, q) = qCS(A, q) + m_{-1}^q$$

where  $m_{-1}^q$  is a constant given by

$$m_{-1}^q = \sum_{\eta \in \pi_0(\mathcal{M}_{n+1,L})} \#(\mathcal{M}_{0,L,\eta}) q^{a_\eta}$$

## Lecture Two

### 2875 Lines on Quintic

**Question:** Given a generic quintic  $X = \{P(x_1, \dots, x_5) = 0\} \subset \mathbb{P}^4$ , where  $\deg P = 5$ , how many lines do we have on  $X$ ?

We write a line as a degree one map

$$\mathbb{P}^1 \rightarrow \mathbb{P}^4, [z, w] \rightarrow [f_i z + g_i w]$$

where  $(f_i, g_i)$  are constants. The above line lies in  $X$  iff

$$f(f_i z + g_i w) = 0, \quad \forall z, w$$

expand it we get

$$P_5(f, g)z^5 + P_4(f, g)z^4w + \dots + P_0(f, g)w^5 = 0$$

where we obtain 6 equations. Note that  $\{(f_i, g_i)\}$  parametrizes the Grassmannian  $G(2, 5)$  which has dimension  $2(5 - 2) = 6$ , and we expect to "count" the solutions. Consider the universal exact sequence of bundles on  $G(2, 5)$

$$\begin{array}{ccccccc} 0 & \longrightarrow & U & \longrightarrow & \mathbb{C}^5 & \longrightarrow & Q \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & G(2, 5) & & \end{array}$$

and it's easy to see that  $P_0, \dots, P_5$  give the sections of  $Sym^5(U^v)$ .  $U^v$  is the dual of  $U$ , which is a rank 6 bundle. Therefore

$$\#\text{lines in } X = c_6(Sym^5(U^v))$$

is just the top chern class. We use splitting principle to calculate the chern class. Assume that

$$c(U) = (1 + a)(1 + b) = 1 + x + y$$

where  $x = c_1(U), y = c_2(U)$ , then

$$c(Sym^5(U^v)) = (1 - 5a)(1 - 4a - b)(1 - 3a - 2b)(1 - 2a - 3b)(1 - a - 4b)(1 - 5b)$$

therefore it's easy to calculate

$$c_6(Sym^5(U^v)) = 25ab(4a + b)(4b + a)(3a + 2b)(2a + 3b) = 25y(24x^4 + 58x^2y + 9y^2)$$

If we write  $c(Q) = 1 + z_1 + z_2 + z_3$  and use the fact that

$$c(U)c(Q) = 1$$

we can obtain

$$y^3 = x^2y^2, x^4 = 2x^2y^2$$

hence

$$c_6(Sym^5(U^v)) = 25(24 * 2 + 9 + 58)x^2y^2 = 2875x^2y^2$$

the number  $x^2y^2$  can be computed by calculating  $c_6(T_{G(2,3)}) = c_6(U^v \otimes Q)$ , and the result is  $x^2y^2 = 1$ . We arrive at the result

$$\#\text{lines on } X = 2875$$

## 15 Real Lines on Real Quintic

Now let  $L$  be a generic real quintic inside  $\mathbb{R}\mathbb{P}^4$ , and we try to compute the number of real lines in  $P$ . We have the same thing except that all the bundles are now real

$$\begin{array}{ccccccc} 0 & \longrightarrow & U_{\mathbb{R}} & \longrightarrow & \mathbb{R}^5 & \longrightarrow & Q_{\mathbb{R}} \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & G_{\mathbb{R}}(2, 5) & & \end{array}$$

Instead of taking the top chern class, we should take the euler class and compute

$$eu(\text{Sym}^5(U_{\mathbb{R}}^v))$$

Let  $U$  be the complexification of  $U_{\mathbb{R}}$ , then now we have

$$c(U) = 1 + y = (1 + a)(1 - a)$$

and

$$c(\text{Sym}^5(U^v)) = (1 - 5a)(1 - 3a)(1 - a)(1 + a)(1 + 3a)(1 + 5a)$$

therefore

$$eu(\text{Sym}^5(U_{\mathbb{R}}^v)) = \sqrt{c(\text{Sym}^5(U^v))} = 15a^3 = 15$$

where  $a^3 = 1$  can be computed similarly. We arrive at

$$\#\text{real lines on } L = 15$$

**Remark.** *Counting real lines on  $L$  is related to counting disks on  $X$  with boundary on  $L$ , where  $X \subset \mathbb{P}^4$  is the quintic with the same defining equation as  $L$ , and we view  $L$  as a Lagrangian submanifold of  $X$ .*

## Physics Origin of Mirror Symmetry

We consider string theory on 10d spacetime, which is a product

$$M^{3,1} \times X$$

where  $M^{3,1}$  is the standard Minkovski spacetime and  $X$  is compact 6-dim manifold. if  $X$  is Calabi-Yau, then using  $\sigma$ -model for perturbation string theory, we obtain 2d QFT with  $N=2$  worldsheet SUSY.

**Physics Question:** Can we reconstruct  $X$  from the 2d QFT itself?

The answer is **No**. It turns out that given a CY 3-fold  $X$ , we can find another CY 3-fold  $Y$  such that

$$\text{physics associated to } X \simeq \text{physics associated to } Y$$

$X$  and  $Y$  are called **mirror manifolds**. One consequence of this equivalence of theories is that counting rational curves on  $X$  is equivalent to some period counting on its mirror  $Y$ .

## Picard-Fuchs Equation

Let  $X_{\psi} = \{W_{\psi} = 0\} \subset \mathbb{P}^4$ , where

$$W_{\psi} = \sum_i \frac{1}{5} x_i^5 - \psi \prod_i x_i$$

The holomorphic 3-form on  $X_{\psi}$  can be represented as residue

$$\Omega_{\psi} = \text{Res}_{W_{\psi}=0} \frac{\omega}{W_{\psi}}, \quad \omega = \sum_i (-1)^i x_i dx_1 \wedge \cdots \hat{d}x_i \wedge \cdots dx_5$$

**Theorem** (Griffith). *Given homogeneous polynomial  $P$  of degree  $5l$ , we have*

$$\text{Res}_{W_\psi} \left( \frac{\omega P}{W_\psi^{l+1}} \right) \subset F^{3-l} H^3$$

here  $F^p H^3 = \bigoplus_{i \geq p} H^{i, 3-i}$  is the Hodge filtration.

We can get a differential equation on  $\Omega_\psi$  using the above relation, which is called **Griffith-Dwork** method. Here's the key observation which is the reduction of poles

$$\frac{l(A^i \partial_i W_\psi) \omega}{W_\psi^{l+1}} = d \left( \frac{A^i \omega_i}{W_\psi^l} \right) - \frac{(\partial_i A^i) \omega}{W_\psi^l}$$

where  $\omega_i = \omega \left( \frac{\partial}{\partial x_i} \right) = \sum (-1)^{i+j} x_j dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge \hat{dx}_j \wedge \cdots \wedge dx_5$ . Here  $A^i$  are arbitrary polynomials with appropriate degree. This reduces the order of pole up to an exact differential form. Using this procedure and the explicit differentiation with respect to  $\psi$ , we obtain

$$\mathcal{L} \left( \frac{\omega}{W_\psi} \right) = d\beta, \quad \mathcal{L} = \theta^4 - 5z(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4), \quad \theta = z \frac{\partial}{\partial z}, z = (5\psi)^{-5}$$

If  $w(\psi) = \int_\Gamma \Omega_\psi$  is a period, where  $\Gamma \in H_3(\mathbb{Z})$  fixed. Then from the above equation we see that the period satisfies the **Picard-Fuchs** equation

$$\mathcal{L}w(\psi) = 0$$

### Moduli of Mirror Quintic

We are in fact doing B-model calculation on the mirror quintic  $Y_\psi = \{W_\psi = 0\}/(\mathbb{Z}_5)^3$ . Its complex moduli is one-dimensional, parametrized by  $\psi$ . Here the group action is given by

$$(\mathbb{Z}_5)^3 = \{x_i \rightarrow e^{\frac{2\pi m_i}{5}} x_i, m_i \in \mathbb{Z}_5 \mid \sum m_i = 0\} / (1, 1, 1, 1, 1)$$

At  $\psi = 0$ , there's an extra  $\mathbb{Z}_5$  symmetry, and this point is called **LG** point. At  $\psi = \infty$ , it's called **large complex limit** point. At  $\psi = 1$ ,  $Y_\psi$  becomes singular, and it's called **conifold** point.

## Lecture Three

### Solutions of Picard-Fuchs Equation

Now we have obtained Picard-Fuchs equation for period integrals

$$\mathcal{L}\omega(z) = 0, \quad \mathcal{L} = \theta^4 - 5z(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4), \quad \theta = z \frac{\partial}{\partial z}, z = (5\psi)^{-5}$$

There's one fundamental solution given by power series

$$\omega_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n$$

The other solutions can be obtained via the following trick: consider the hypergeometric generating function

$$\omega_0(z, H) = \sum_{n=0}^{\infty} \frac{\Gamma(1 + 5(n + H))}{\Gamma(1 + n + H)^5} z^{n+H}$$

where  $\Gamma$  is the Gamma function. Then we check that

$$\mathcal{L}\omega_0(z, H) = H^4 \frac{\Gamma(1 + 5H)}{\Gamma(1 + H)^5} z^H$$

It follows that the solutions are given by

$$\omega_i(z) = \left( \frac{\partial}{\partial H} \right)^i \omega_0(z, H)|_{H=0}, \quad 0 \leq i \leq 3$$

The **mirror map** is given by

$$e^{2\pi it} = q = \exp\left(\frac{\omega_1(z)}{\omega_0(z)}\right) = z + \dots$$

where  $z$  is viewed as complex moduli on  $Y$ , while  $t$  is the complexified kahler moduli on  $X$ . As  $z \rightarrow 0$  the large complex limit point of  $Y$ , we see that  $t \rightarrow i\infty$  which is the large volume limit of  $X$ . The mirror map identifies two neighborhoods of large limits as we want.

### Yukawa Coupling

On the A-model side, Yukawa coupling is the generating function for curve counting (GW invariants)

$$K_{ttt}(q) = 5 + \sum_{d=1}^{\infty} n_d^{(0)} d^3 q^d, \quad q = e^{2\pi it}$$

where  $n_1^{(0)} = 2875$  is the number of lines on the quintic.

On the B-model side, Yukawa coupling is given by variation of hodge structure

$$C_{\psi\psi\psi} = \int_Y \Omega \wedge (\partial_{\psi})^3 \Omega$$

where  $\partial_\psi$  is the Gauss-Manin connection,  $\Omega$  is the holomorphic 3-form on  $Y_\psi$ , with its explicit dependence on  $\psi$  as above. To compute  $C_{\psi\psi\psi}$ , we consider the following relation

$$0 = \int \Omega \wedge (\partial_\psi)^2 \Omega$$

This is because we have in general the Griffith transversality:

$$\partial_\psi \Omega \in F^2 H^3, (\partial_\psi)^2 \Omega \in F^1 H^3$$

and the above integral vanishes by the type reason. Take differentiation with  $\psi$ , we get

$$\int \partial_\psi \Omega \wedge (\partial_\psi)^2 \Omega + \int \Omega \wedge (\partial_\psi)^3 \Omega = 0$$

Take differentiation with  $\psi$  again, and use the relation  $\int (\partial_\psi)^2 \Omega \wedge (\partial_\psi)^2 \Omega = 0$  (since  $(\partial_\psi)^2 \Omega$  is 3-form), we get

$$\partial_\psi C_{\psi\psi\psi} = \frac{1}{2} \int \Omega \wedge (\partial_\psi)^4 \Omega$$

Now we plug in Picard-Fuchs equation and obtain

$$\partial_\psi C_{\psi\psi\psi} = \frac{5\psi^4}{1-\psi^5} C_{\psi\psi\psi}$$

which can be solved by

$$C_{\psi\psi\psi} = \frac{C_0}{1-\psi^5}$$

here  $C_0$  is a constant to be determined later. Now after change of variable  $\psi \rightarrow t$ , and identifying the two Yukawa couplings, we get

$$K_{ttt} = \left(\frac{\partial\psi}{\partial t}\right)^3 \frac{1}{\omega_0^2} C_{\psi\psi\psi} = \left(\frac{\partial\psi}{\partial t}\right)^3 \frac{1}{\omega_0^2} \frac{C_0}{1-\psi^5}$$

Using the computation of number of lines on the quintic,  $K_{ttt} = 5 + 2875q + \dots$ , this can be used to fix the constant  $C_0$ , and then the B-model Yukawa coupling gives the prediction for all counting of rational curves  $n_d^{(0)}$  on the A-model side.

**Remark.** *The prepotential is given by*

$$F^{(0)} = \frac{5}{6}t^3 + c_2 t^2 + c_1 t + \frac{\zeta(3)}{\pi^3} + \sum n_d q^d$$

*This is the holomorphic function that determines the couplings in  $N=2$  theory. Its existence implies that the moduli space is special Kahler manifold.*

## Monodromy

Here're some remarks on the monodromy around the degenerate points.

1. Large complex limit point. The monodromy matrix around this point looks like

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is maximal unipotent.

2. Conifold point. The monodromy around this point can be normalized as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

therefore in the four linearly independent solutions of periods, three of them are given by power series solutions. We don't know how to analytically extend these guys to those near the large complex limit point.

3. LG point. Near this point, the monodromy can be normalized by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}$$

The mirror of periods around this point predicts A-model Orbifold Gromov-Witten invariants.

## SUSY QM

In SUSY QM, the Hilbert space is  $\mathbb{Z}_2$ -graded

$$\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$$

with Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}(h'(x))^2 + \frac{1}{2}h''(x)(\psi^\dagger\psi - \psi\psi^\dagger)$$

here  $h$  is polynomial potential. We have the commutative relation

$$[p, x] = -i, \quad \{\psi, \psi^\dagger\} = 1$$

which can be realized as

$$p = -i\frac{\partial}{\partial x}, \dots$$

We have the odd symmetry  $Q, Q^\dagger$  satisfying

$$\begin{aligned} Q^2 = 0, (Q^\dagger)^2 = 0 \\ [Q, H] = 0, H = \frac{1}{2}\{Q, Q^\dagger\} \end{aligned}$$

explicitly

$$Q = \psi^\dagger(ip + h'(x)), \quad Q^\dagger = \psi(-ip + h'(x))$$

### Representation of SUSY Algebra

In the 1-dim case,

$$Q = Q^\dagger = H = 0$$

In the 2-dim case,

$$Q = \begin{pmatrix} 0 & \sqrt{2E} \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ \sqrt{2E} & 0 \end{pmatrix}, \quad H = E$$

## SUSY Ground State

The ground state is annihilated by  $Q, Q^\dagger$ . Let  $\phi = (\phi^{(0)}(x), \phi^{(1)}(x))$ , then

$$(ip + h')\phi^{(0)} = 0, (-ip + h')\phi^{(1)} = 0$$

which is solved by

$$\phi^{(0)} = \alpha e^{-h(x)}, \phi^{(1)} = \beta e^{h(x)}, \quad \alpha, \beta \text{ constants}$$

If we write  $h(x) = a_d x^d + \text{lower degree}$  and we assume that  $d$  is even. Then only one of  $\phi^{(0)}$  and  $\phi^{(1)}$  is renormalizable at  $\infty$ . We get 1-dimensional SUSY ground state.

**Remark.** *The situation is different if we consider the complex case, where  $z = x + iy, \psi_z = \psi_x + i\psi_y$  and  $h = h(z)$  holomorphic, then the number of ground state equals the number of critical points of  $h$ , one less than the degree of  $h$ . While in the real case above, the number of SUSY ground state is one if  $d$  is even and zero when  $d$  is odd.*

## SUSY QM on Riemannian Manifold

Let  $(M, g)$  be Riemannian manifold, the field is given by

$$\phi : \mathbb{R} \rightarrow X, \psi \in \Gamma(\phi^*(TX) \otimes \mathbb{C})$$

The Lagrangian is given by

$$L = \frac{1}{2}g(\dot{\phi}, \dot{\phi}) + \frac{i}{2}g(\bar{\psi}, D_{\dot{\phi}}\psi) - \frac{i}{2}g(D_{\dot{\phi}}\bar{\psi}, \psi) - \frac{1}{2}g(R(\psi, \bar{\psi})\psi, \bar{\psi})$$

the SUSY is given by

$$\begin{aligned} \delta\phi &= \bar{\psi} & \delta\phi^I &= \bar{\psi}^I \\ \delta\psi &= i\dot{\phi} - D_{\bar{\psi}}\psi & \delta\psi^I &= i\dot{\phi}^I - \Gamma_{JK}^I \bar{\psi}^J \psi^K \\ \delta\bar{\psi} &= 0 \end{aligned}$$

Then the homework problem is to check that

$$\delta L = 0$$

Let's do the variation:

$$\begin{aligned} \delta \frac{1}{2}g(\dot{\phi}, \dot{\phi}) &= g(\dot{\psi}, \dot{\phi}) + g(D_{\bar{\psi}}\dot{\phi}, \dot{\phi}) = g(D_{\dot{\phi}}\bar{\psi}, \dot{\phi}) \\ \delta \left( \frac{i}{2}g(\bar{\psi}, D_{\dot{\phi}}\psi) \right) &= \frac{i}{2}g(D_{\bar{\psi}}\bar{\psi}, D_{\dot{\phi}}\psi) - \frac{i}{2}g(\bar{\psi}, D_{\bar{\psi}}D_{\dot{\phi}}\psi) - \frac{1}{2}g(\bar{\psi}, D_{[\dot{\phi}, \bar{\psi}]\psi}) - \frac{1}{2}g(\bar{\psi}, D_{\dot{\phi}}(i\dot{\phi} - D_{\bar{\psi}}\psi)) \\ &= -\frac{1}{2}g(D_{\dot{\phi}}\bar{\psi}, \psi) + \frac{i}{2}g(R(\dot{\phi}, \bar{\psi})\bar{\psi}, \psi) \\ \delta \left( -\frac{i}{2}g(D_{\dot{\phi}}\bar{\psi}, \psi) \right) &= -\frac{1}{2}g(D_{\dot{\phi}}\bar{\psi}, \dot{\phi}) + \frac{i}{2}g(R(\dot{\phi}, \bar{\psi})\bar{\psi}, \psi) \\ \delta \left( -\frac{1}{2}g(R(\psi, \bar{\psi})\psi, \bar{\psi}) \right) &= -\frac{1}{2}g(D_{\bar{\psi}}(R(\psi, \bar{\psi})\psi), \bar{\psi}) - \frac{1}{2}g(R(i\dot{\phi} - D_{\bar{\psi}}\psi, \bar{\psi})\psi, \bar{\psi}) - \frac{1}{2}g(R(\psi, \bar{\psi})(i\dot{\phi} - D_{\bar{\psi}}\psi), \bar{\psi}) \\ &= -\frac{1}{2}g((D_{\bar{\psi}}R)(\psi, \bar{\psi})\psi, \bar{\psi}) - \frac{i}{2}g(R(\dot{\phi}, \bar{\psi})\psi, \bar{\psi}) - \frac{i}{2}g(R(\psi, \bar{\psi})\dot{\phi}, \bar{\psi}) \end{aligned}$$

Sum them together we get

$$\delta L = \frac{1}{2}g((D_{\bar{\psi}}R)(\psi, \bar{\psi})\psi, \bar{\psi}) = 0$$

by Bianchi-Identity. The supercharges are obtained via Noether method

$$Q = ig(\dot{\phi}, \bar{\psi}), \quad Q^\dagger = ig(\dot{\phi}, \psi)$$

Therefore we have  $N = 1$  SUSY on Riemannian manifold.

## N=2 SUSY on Kahler Manifold

If the target manifold  $X$  is Kahler, then the tangent bundle splits into holomorphic and anti-holomorphic parts

$$TX \otimes \mathbb{C} = TX^h \oplus T^{\bar{h}}X$$

$$Q = ig(\phi, \bar{\psi}^h) + ig(\phi, \bar{\psi}^{\bar{h}}) = Q^h + Q^{\bar{h}}$$

and  $Q^h, Q^{\bar{h}}$  are separately preserved. The SUSY ground state corresponds to cohomology of  $X$ , with additional symmetry called R-symmetry, which in the Kahler case gives the Hodge decomposition.

### 2d $\sigma$ -model

Let  $\Phi$  be a map of supermanifold

$$\Phi : \mathbb{R}^{2|4} \rightarrow X$$

with Lagrangian given by superpotential

$$L = \int d^4\theta K(\bar{\Phi}, \Phi)$$

### LG Model

In LG model, we need non-compact target with holomorphic function  $W$  on it. The Lagrangian is given by

$$L = \int d^4\theta \bar{\Phi} \Phi + \int d^2\theta W + \int d^2\bar{\theta} \bar{W}$$

### Global Picture of Mirror Symmetry

We consider complete intersection in toric varieties. For example the Quintic

$$X = \{W = 0\} \subset \mathbb{P}^4$$

and its mirror

$$Y = \{W = 0\} / \mathbb{Z}_5^3$$

On the A-model side, we have the picture

$$t = i\infty : \text{Kahler} \xleftrightarrow{\text{CY/LG correspondence}} t = -i\infty : \text{LG Model on } \mathbb{C}^5 / \mathbb{Z}_5 \text{ with superpotential } W$$

On the B-model side, we also have

$$\psi = 0 : \text{Large Complex} \xleftrightarrow{\text{CY/LG}} t = \infty : \text{LG point}$$

and we have the corresponding mirror symmetries for A and B models.

## Lecture Four

### Counting Disk

Let  $X$  be real Quintic ("real" means the defining equation has real coefficient),  $L$  be the real locus of its defining equation. Let  $n_d$  be the GW invariant counting

$$(D, \partial D) \rightarrow (X, L)$$

and consider the generating function

$$T(q) = \frac{\log q}{2} \pm \frac{1}{4} \pm \sum_{d \text{ odd}} n_d q^{d/2}$$

where  $d \in H_2(X, L) \simeq \mathbb{Z}$ . Note that we have exact sequence

$$H_2(X) \rightarrow H_2(X, L) \rightarrow H_1(L)$$

and  $H_1(L) = \mathbb{Z}_2$ . Therefore  $d$  odd means that the boundary circle is nontrivial in  $L$ . The mirror of  $T(q)$  has Hodge theoretical interpretation by normal functions.  $L = \mathbb{RP}^3$  has two flat connections.

In physics,  $T$  is the domainwall tension between two  $N=1$  vacua corresponding to  $(L, \pm)$ . Recall that the tension has in general the structure: classical+instantons. The tension of  $D_6$  wrapping on 4-cycle is given by

$$\int_{C^4} \omega \wedge \omega + \sum n_d^{(0)} q^d$$

where  $t = \log q$ . The tension of  $D_4$  brane wrapping on 2-cycle is given by

$$\int_{C^{(2)}} \omega = t$$

The term  $\frac{\log q}{2}$  is the classical tension of the domainwall (where we have  $t = \log q$ ), and the term  $\frac{1}{4}$  is added to be compatible with the monodromy operation  $t \rightarrow t + 1$ .

In A-model, we indeed have 625 real lagrangians, and 625 antiholomorphic evolutions:

$$x_i \rightarrow \omega_i \bar{x}_i, \quad (\omega_i)^5 = 1$$

therefore we get 625 pairs of objects  $L_{\pm}^{[\omega]}$  in the Fukaya category. The objects that are mirrored to  $L_{\pm}^{[\omega]}$  are matrix factorization of

$$W = \sum \frac{1}{5} x_i^5 - \psi x_1 \cdots x_5$$

Using CY/LG correspondence (theorem of Orlov, Herbst-Hori-Pags), we have

$$D^b(Y_{\psi}) = MF(W_{\psi}, \mathbb{Z}_5^4)$$

This is obtained by GLSM. For example, here we have  $U(1)$  charges  $(1,1,1,1,1,-5)$  to  $(x_1, x_2, x_3, x_4, x_5, P)$  with superpotential  $G = PV$ . The vacuum manifold is determined by

$$\left\{ \sum |x_i|^5 - 5|P|^2 = \text{Re } t \right\} / U(1)$$

if  $\text{Re } t \gg 0$ , then we get the critical point which is

$$X = \{P = V = 0\} \subset \mathcal{O}_{\mathbb{P}^4}(-5)$$

If we consider  $\text{Re } t \ll 0$ , we get LG on  $(\mathbb{C}^5/\mathbb{Z}_5, V)$ . Now if we study D-branes in GLSM, then we have the picture

$$\begin{aligned} \text{Re } t \gg 0 & : D^b(X) \\ \text{Re } t \ll 0 & : MF(V, \mathbb{Z}_5) \end{aligned}$$

### Matrix Factorizations

$W$  is a polynomial, then a matrix factorization is a pair of matrix  $f, g$  with polynomial entries such that

$$fg = gf = W \cdot id$$

Usually we write

$$Q = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}$$

then  $Q^2 = id$ .  $MF(W)$  is the category with objects the matrix factorizations above, with morphism

$$H^*(Mat_{2N \times 2N'}(\mathbb{C}[x]), D)$$

Here given  $Q_{2N \times 2N}, Q'_{2N', 2N'}$ , we can define

$$D\Phi = Q\Phi - (-1)^\Phi \Phi Q'$$

where

$$(-1)^\Phi = \sigma\Phi\sigma', \quad \sigma = \begin{pmatrix} id_N & 0 \\ 0 & -id_N \end{pmatrix}, \quad \sigma' = \begin{pmatrix} id_{N'} & 0 \\ 0 & -id_{N'} \end{pmatrix}$$

We also require the  $\mathbb{Z}_5$ -equivariance, which says that we need to consider  $2N \times 2N$  representations of  $\mathbb{Z}_5$  which under  $(x_1, \dots, x_5) \rightarrow (\omega x_1, \dots, \omega x_5)$  satisfies

$$\rho(\omega)Q(\omega x_i)\rho^{-1}(\omega) = Q(x_i)$$

The mirrors of  $L_{\pm}^{[\omega]}$  is

$$Q_{\pm} = \sum_{i=1}^5 \frac{1}{\sqrt{5}} (x_i^2 \eta_i + x_i^3 \bar{\eta}_i) \pm \sqrt{\psi} \prod_{i=1}^5 (\eta_i - x_i \bar{\eta}_i)$$

where

$$\{\eta_i, \bar{\eta}_j\} = \delta_{ij}$$

then we check that  $Q_{\pm}^2 = W$ . There are 625 ways to make  $Q_{\pm}$   $\mathbb{Z}_5^4$  equivariant, which corresponds to the 625 real lagrangians. There's no machinery to write down a matrix factorization explicitly corresponding to elements in Fukaya category, and the above one is the conjectured objects

**Remark.** *At the fermat point  $\psi = 0$ , we have*

$$MF(\sum x_i^5) = \otimes_{i=1}^5 MF(x_i^5)$$

*and each  $MF(x_i^5, \mathbb{Z}_5)$  is easy to understand, essentially there're only two*

$$\begin{pmatrix} 0 & x \\ x^4 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & x^2 \\ x^3 & 0 \end{pmatrix}$$

## B-model

The B-model tension is given by the difference of holomorphic chern-simons valued in different critical point.

$$T_{+|-} = hCS(E_+) - hCS(E_-)$$

where

$$hCS(A) = \int_Y \Omega \wedge \text{Tr} \left( A \bar{\partial} A + \frac{2}{3} A^3 \right)$$

whose critical points corresponds to holomorphic bundle. This should be the mirrors of Fukaya category with ordinary CS and disk instanton. This is possibly only true around the critical points, not the general off-shell deformations.

Using CY/LG correspondence for B-branes, we can obtain two objects  $E_{\pm} \in D^b(Y)$  from two matrix factorizations  $Q_{\pm}$ , and the domainwall tension turns out to be

$$hCS(E_+) - hCS(E_-) = \int_{\Gamma} \Omega$$

where  $\Gamma$  is a 3-cycle whose boundary is exactly

$$\partial\Gamma = C_+ - C_-, \quad C_{\pm} = c_2^{alg}(E_{\pm})$$

where  $c_2^{alg}$  is the algebraic second chern class which lies in chow group  $CH^2(Y)$ . Note that the well-definedness of the domainwall tension as integrals follows from Hodge theory (normal functions). After some computations, we will obtain

$$C_{\pm}^{[\omega, \omega']} = \{x_1 + \omega x_2 = 0, x_3 + \omega' x_4 = 0, x_5^2 \pm \sqrt{5\psi} \omega \omega' x_1 x_3 = 0\}$$

where  $\omega, \omega' \in \mathbb{Z}_5$ , and

$$C_{\pm} = \bigcup_{\omega, \omega'} C_{\pm}^{[\omega, \omega']}$$

we denote the domainwall tension by

$$T(z) = \int_{C_-}^{C_+} \Omega(z)$$

To compute T, let

$$\mathcal{L} = \theta^4 - 5z(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4), \quad \theta = z \frac{\partial}{\partial z}$$

and use the relation obtained from Griffith-Dwork reduction

$$\mathcal{L}\Omega = d\beta$$

we can compute

$$\mathcal{L}T(z) = \int_{C_-}^{C_+} d\beta = \int_{C_+} \beta - \int_{C_-} \beta$$

the RHS can be computed in a brute-force way and at the end of day we would obtain

$$\mathcal{L}T(z) = \frac{15}{\pi^2} z^{\frac{1}{2}}$$

Note that 15 is just the number of real lines on real quintic!

## Lecture Five

### Normal Functions and Infinitesimal Invariants

Let's study the situation of relative period more carefully. Let  $C_{\pm}$  be two holomorphic curves as before,  $\Gamma$  is three cycle such that

$$\partial\Gamma = C_+ - C_-$$

and consider

$$\int_{C_-}^{C_+} \Omega = \int_{\Gamma} \Omega$$

Since  $\Omega$  is only a cohomology class, and  $\Gamma$  is not a closed cycle. We need to check that it's well-defined. In fact, the integral takes value in

$$\int_{C_-}^{C_+} \in (F^2 H^3)^* / H_3(Y, \mathbb{Z}) = H^3(Y) / (F^2 H^3 + H^3(Y, \mathbb{Z})) = J^2(Y)$$

which is called Intermediate Jacobian.

In families over  $M$  the moduli space of complex structure, we have intermediate Jacobian fibration

$$J^2(Y) \rightarrow M$$

Normal function is defined to be holomorphic section  $\nu$  of  $J^2(Y)$  satisfying Griffiths transversality, i.e.,

$$\nabla_z \tilde{\nu} \in F^1 H^3$$

here  $\tilde{\nu}$  is any local lifting of  $\nu$  to  $H^3(Y, \mathbb{C})$ , and  $\nabla_z$  is the GM connection. As a consequence, we only need the truncated normal function (superpotential)

$$T = \nu(\Omega), \quad \Omega \in H^{3,0}$$

then

$$\partial_z T = (\nabla_z \nu)(\Omega) + \nu(\nabla_z \Omega)$$

and  $(\nabla_z \nu)\Omega = 0$  by the property of normal function.

**Definition.** *Griffiths Infinitesimal invariant* of normal function is defined to be

$$\Delta_{zz} = \nabla_z^2 \tilde{\nu}(\Omega) = -\nabla_z \tilde{\nu}(\nabla_z \Omega)$$

*Note that this in general depends on the lifting  $\tilde{\nu}$ . We choose a **real** lifting of  $\tilde{\nu}$  which means that*

$$\tilde{\nu}(\bar{\Omega}) = \overline{\tilde{\nu}(\Omega)}$$

*The real lifting is defined upto integral period, which disappears after taking derivatives in the definition of  $\Delta_{zz}$ .*

Note that  $\Delta_{zz}$  is not holomorphic. We claim that

$$\bar{\partial}_{\bar{z}} \Delta_{zz} = -C_{zzz} \bar{\Delta}_{\bar{z}\bar{z}} G^{z\bar{z}} e^K$$

this is the holomorphic anomaly for infinitesimal invariants. In fact, using

$$\Delta_{zz} = D_z^2 T - C_{zzz} G^{z\bar{z}} e^K D_{\bar{z}} \bar{T}$$

where  $D_z$  is the covariant derivative on  $H^3(Y)$  using the kahler metric with Kahler potential

$$e^K = \int \Omega \wedge \bar{\Omega}$$

Using special geometry relation

$$[\bar{D}_{\bar{a}}, D_z]_{\bar{z}} = C_{zzz} \bar{C}_{\bar{z}\bar{z}}^z$$

the check of holomorphic anomaly equation above then goes roughly as follows

$$\begin{aligned} \bar{\partial} \Delta_{zz} &= [\bar{\partial}_z, D_z] D_z T - C_{zz}^{\bar{z}} \bar{D}_{\bar{z}}^2 \bar{T} \\ &= C \bar{C} D_z T - C \bar{D}_{\bar{z}}^2 \bar{T} \\ &= -C (\bar{D}_{\bar{z}} \bar{T} - \bar{C} D_z T) \\ &= -C \bar{\Delta}_{\bar{z}\bar{z}} \end{aligned}$$

**Remark.** *Relation between extended PF: In one-parameter case, in special coordinate,*

$$\mathcal{L} = \partial_t^2 \frac{1}{K_{ttt}} \partial_t^2$$

*Inhomogeneous PF in this coordinate is given by*

$$\partial_t^2 \frac{1}{K_{ttt}} \partial_t^2 T = \partial_t^2 \frac{\Delta_{tt}}{K_{ttt}}$$

### Holomorphic Anomaly Equation

We consider topological string amplitude  $F^{(g)}$ ,  $g \geq 0$ . On the A-model side,

On the B-model side, we have

$$F_B^{(0)} = \frac{1}{2} \omega^A \omega_A(z)$$

which is holomorphic. But for higher genus  $F_B^{(g)}$  is no longer holomorphic, and satisfies the holomorphic anomaly equation

$$\bar{F}^{(g)} = \frac{1}{2} \bar{C}_{\bar{z}\bar{z}}^{zz} \sum_{g_1+g_2=g} \left( D_z F^{(g_1)} D_z F^{(g_2)} \right) + \frac{1}{2} \bar{C}_{\bar{z}\bar{z}}^{zz} D_z^2 F^{(g-2)}$$

This anomaly equation comes from contribution from "boundary" of  $M_g$ :  $\bar{\partial} \int_{M_g} \langle \dots \rangle$  with some measure on  $M_g$ .

Now we consider the Riemann surfaces of genus  $g$  and  $h$  boundary components. We have similarly defined open string amplitude

$$F^{(g,h)} = \int_{M^{(g,h)}} \langle \dots \rangle$$

The case  $(g, h) = (0, 1)$  is captured by normal function  $T$ . The extension of holomorphic anomaly is given by

$$\bar{\partial}_{\bar{z}} F^{(g,h)} = \frac{1}{2} C_{\bar{z}\bar{z}}^{zz} \sum_{g_1+g_2=g, h_1+h_2=h} D_z F^{(g_1, h_1)} D_z F^{(g_2, h_2)} + \frac{1}{2} \bar{C}_{\bar{z}\bar{z}}^{zz} D_z^2 F^{(g-1, h)} - \bar{\Delta}_{\bar{z}\bar{z}}^z D_z F^{(g, h-1)}$$

**Remark.**  $F^{(0,h)}$ : It seems that there's a canonical way to fix the holomorphic ambiguity for this guy, while for  $F^{(g,0)}$  we don't have a canonical way to fix the ambiguity.

**Remark.**  $\Delta_{zz}$  is 2 closed string insertions on disk, while  $C_{zzz}$  is 3 closed string insertions on the sphere.

**Remark.** Let's look at the one-loop amplitude.

$$\bar{\partial}_{\bar{z}}\partial_z F^{(0,2)} = -\Delta_{\bar{z}}^z \Delta_{zz} + G_{z\bar{z}}$$

this is not well-understood from the mathematical point of view.