On integral Stokes matrices

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Frobenius manifolds

A Frobenius manifold is a tuple $(M, \circ, g, E)$ where

M is a (complex) manifold

$\circ$ is a product on TM

g is a flat metric

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We are interested in the (generically) semisimple case.

Main sources of examples

Quantum cohomology for Fano manifolds (e.g. projective spaces, Grassmannians, ...)

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Geometry of Hurwitz spaces (Modular Frobenius manifolds. In dim 3 this is related to Chazy equations)

First structure connection

The first structure connection of a Frobenius manifold is a flat connection with singularities on the pullback p^*TM of the tangent bundle to $\mathbb{P}^1 \times M$ via the projection $p : \mathbb{P}^1 \times M \to M$ given by

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Restricting to any fibre $\{m\} \times \mathbb{P}^1$ we get a connection on $\mathbb{P}^1$ with singularities at 0 (regular point) and at ∞ (irregular point of order 2)

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$$E\circ = \begin{pmatrix} 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & k \\ kq & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

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$$\mu = \mathrm{diag}\left(\tfrac{k-1}{2}, \tfrac{k-3}{2}, \cdots, -\tfrac{k-3}{2}, -\tfrac{k-1}{2}\right)$$

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2.b. What is the meaning of the integrality?

Cecotti-Vafa in the 90's (studying the so-called tt^*-equations) showed that the classification of $\mathcal{N} = 2$ supersymmetric models is reduced to the problem:

Find all integral strictly upper triangular matrices A of size $n \times n$ such that all the eigenvalues λ_i of the matrix $(1 - A)(1 - A^T)^{-1}$ belong to the unit circle.

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An important problem is to study solutions of certain equations (known as Toda lattice with opposite sign) whose associated $\frac{\infty}{2}$ variations of Hodge structures have $\mathbb{Z}$-structure.

What we need is to compute the Stokes matrices (associated to those tt^*-equations) and show the integrality (problem 1 and problem 2a)

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Today I will discuss how to make these computations by the so-called monodromy identity method.

Formal classification of differential modules

Theorem (Levelt-Turrittin)

Let $\mathcal{M}$ be a differential module of finite dimension. There exists a finite field extension $\hat{K}_b$ of $\hat{K} = \mathbb{C}((z))$ and there are distinct elements $q_1, \cdots, q_s \in z^{-\frac{1}{b}}\mathbb{C}[z^{-\frac{1}{b}}]$ such that $\hat{K}_b \otimes \mathcal{M}$ decomposes as

$$\bigoplus_{i=1}^{s} E(q_i) \otimes N_i$$

where $E(q_i)$ is the one dimensional module $\hat{K}.e_{q_i}$ with $\partial_z e_{q_i} = q_i e_{q_i}$ and N_i is a regular singular differential module over $\hat{K}$

b is called the ramification index

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The equation coming from the first structure connection of a Frobenius manifold is unramified.

Definition

A Frobenius manifold is called $\mathbb{Z}_n$-symmetric if the eigenvalues of the matrix multiplication by the Euler vector field are $n - th$ roots of unity.

This terminology comes from the relation between the Frobenius manifolds with the given property of the eigenvalues and the $\mathbb{Z}_n$-symmetric models studied by Cecotti-Vafa.

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Proposition (CM)

If A Frobenius manifold is $\mathbb{Z}_n$-symmetric then there exists a change of variable such that the associated differential equation for the first structure connection becomes totally ramified

Monodromy identity

Consider $q_i - q_j = c_0 + c_1 z^{-1} + \cdots + c_k z^{-k}$, $c_k \neq 0$, $i \neq j$. A **Stokes direction** $d \in \mathbb{R}(\mathrm{mod} 2\pi\mathbb{Z})$ for (i, j) is a direction such that $\mathrm{Re}(\frac{c_k}{k} z^{-k}) = 0$ for $z = \mid z \mid e^{\sqrt{-1}d}$.

$d = \frac{1}{k}\arg(c_k) + \frac{\pi}{2}(s + \frac{1}{2})$, $s = 0, 1, \cdots, 2k - 1$

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For any singular direction d we get the Stokes matrix (Stokes factor) St_d

Monodromy identity

Theorem

Let $d_1 < d_2 < \cdots < d_m$ be the singular direction for the collection $\{q_i - q_j\}$. Then $MStd_{d_m} \cdots St_{d_1}$, where M is the formal monodromy, is conjugate to the topological monodromy

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In our case the topological monodromy is the monodromy at 0 (singular regular point)

Proposition (van der Put, van der Put - CM)

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Theorem (Mochizuki, van der Put-CM, CM)

The equation coming from the first structure connection of a $\mathbb{Z}_n$*-symmetric Frobenius manifold has an integral structure if and only if the coefficients of the characteristic polynomial of the topological monodromy along* $z = 0$ *are integer.*

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Conjecture (Very vague conjecture)

For a (generically) semisimple Frobenius manifold with an integral structure there exists a derived category (of representations of certain quiver) such that the Stokes matrix associated to the first structure connection equals the Gramm matrix for a full exceptional collection.

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What is the role of Dubrovin's Landau-Ginzburg superpotential associated to (semisimple) Frobenius manifolds?

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How about Γ-conjectures?

A deeper study of the integral structure should be done in the framework of (variations of) non-commutative Hodge structures in the sense of Katzarkov-Kontsevich-Pantev (a generalization of Hodge structures that includes equations of the type obtained from the first structure connections of semisimple Frobenius manifolds)

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How about the geometricity for variations of noncommutative Hodge structures?