## Context: where Lie algebras meet field theory

**Model situation:**

$Ug$ as "edge currents" for the 2d gauge theory known as topological BF theory

**Recall:** char 0 field $k = \mathbb{R}$ or $\mathbb{C}$

Every assoc. algebra has an "underlying" algebra

$$\mathrm{Alg}_{\mathrm{Ass}} \xrightarrow{\text{"forget"}} \mathrm{Alg}_{\mathrm{Lie}}$$

$$(A, \star) \longmapsto (A, [\,,\,])$$

$$[a,b] = a \star b - b \star a$$

There's a "dual" construction: the left adjoint

$$\mathrm{Alg}_{\mathrm{Lie}} \xrightarrow{U} \mathrm{Alg}_{\mathrm{Ass}}$$

$$\mathfrak{g} \longmapsto U\mathfrak{g} = \mathrm{Tens}(\mathfrak{g}) \big/ (x \otimes y - y \otimes x = [x,y])$$

universal enveloping algebra

This is obviously important in algebra but it also shows up in quantum mechanics:

$\mathfrak{g} \circlearrowright$ quantum mechanical system

symmetries

$$\rightsquigarrow \quad U\mathfrak{g} \xrightarrow{J} A = \text{algebra of operators for system}$$

A particularly nice situation is when A arises as the deformation quantization of some Poisson manifold $X$.

Here $A$ is a deformation of $\mathcal{O}(X)$.

Note that $U\mathfrak{g}$ is itself a def. quantization of $\mathrm{Sym}(\mathfrak{g}) = \mathcal{O}(\mathfrak{g}^*)$ with the Poisson bracket determined by Lie bracket on $\mathfrak{g}$.

This means $J$ "dequantizes" to a Poisson map

$$J^{cl} : \underset{\mathcal{O}(\mathfrak{g}^*)}{\underset{\shortparallel}{\mathrm{Sym}(\mathfrak{g})}} \longrightarrow \mathcal{O}(X)$$

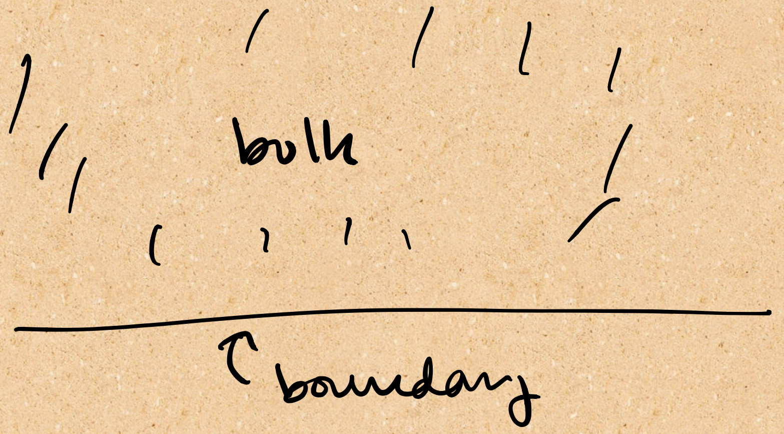
which is equivalent to specifying a moment map

$$X \longrightarrow \mathfrak{g}^*.$$

But there's another way that $U\mathfrak{g}$ shows up in 2d field theory.

Fix a Lie algebra $\mathfrak{g}$.

Consider the "spacetime"




$\mathbb{H} = \mathbb{R} \times \mathbb{R}_{\geq 0}$

$\mathbb{R} \times \{0\}$

In the bulk, we place topological BF theory with fields

$$A \in \Omega^1(\mathring{\mathbb{H}}) \otimes \mathfrak{g}.$$

connection 1-form

$$B \in \Omega^0(\mathring{\mathbb{H}}) \otimes \mathfrak{g}^*.$$

and action functional

$$S_{BF}(A,B) = \int_M (B, F_A)$$

$F_A = dA + \frac{1}{2}[A,A]$

The equations of motion are

$$\begin{cases} F_A = 0 & \leftarrow \text{flat connections} \\ d_A B = dB + [A,B] & \leftarrow B \text{ cov. constant} \end{cases}$$

so

$$\mathrm{Sol} \simeq T^*[1]\,\mathrm{Flat}_G$$

cotangent direction is B

moduli of flat G-connections

For a region U in the bulk


the observables are

$$\mathrm{Obs}^{cl}(U) \simeq \mathcal{O}(\mathrm{Sol}(U))$$
$$\simeq \mathcal{O}(T^*[1]\,\mathrm{Flat}_G(U))$$

We impose the "boundary condition" that $A\big|_{\partial H} = 0$ but $B$ is free.

Hence we're asking that along $\partial H$, $dB = 0$ so it's constant so a solution $B \in \mathfrak{g}^*$

$\rightsquigarrow$ 2dk

V

$$\mathrm{Obs}^{cl}(V) \simeq \mathcal{O}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g})$$

<u>Claim</u> You can perturbatively quantize this bulk-boundary system so that

$$\mathrm{Obs}^q(V) \simeq U\mathfrak{g}$$

bdry opns

$U\mathfrak{g}$ as "edge currents"

and

$$\mathrm{Obs}^q(U) \simeq \mathrm{Hoch}^*(U\mathfrak{g}, U\mathfrak{g})$$

bulk opns

physics

algebra

Rmk This claim is part of body of work that grew out of Kontsevich's def. quantization results.

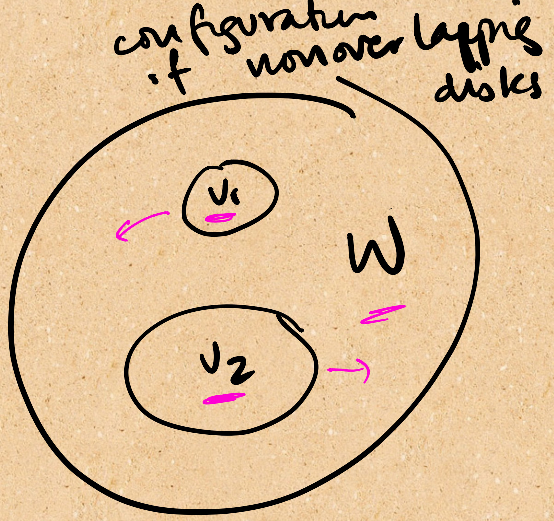
<u>Aside</u> : Eugene Rabinovich (graduating student at Berkeley) is developing a version of Costello's machinery that encompasses such bulk-bdry systems

——— end of model ———

<u>Today</u>: Describe joint work w/ Ginot, Williams, Zeinalian tackling such situations via the kind of universal properties that $U\mathfrak{g}$ has

# Bulk algebras

In the bulk, the observables Obs has the structure of <u>an algebra over the little 2-disks operad</u> $\mathcal{E}_2$




$$\begin{array}{c} \mathrm{Obs}(U_1) \otimes \mathrm{Obs}(U_2) \\ \downarrow \\ \mathrm{Obs}(W) \end{array}$$

There is a systematic generalization to an operad

$$\mathcal{E}_n = \text{little } n\text{-disks operad}$$

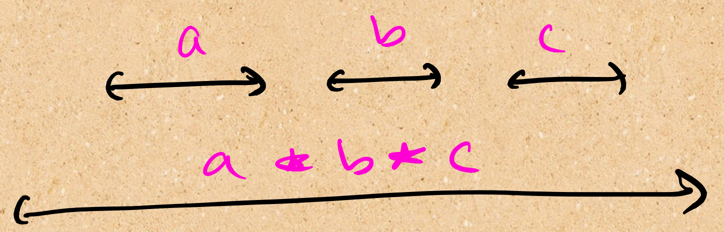
where multiplications are parametrized by configurations of disjoint $n$-dim disks in $\mathbb{R}^n$

As a special case $n=1$, it's helpful to know

$$\underbrace{\mathrm{Alg}_{\mathcal{E}_1}(\mathrm{Ch}_k^{\otimes})}_{\mathcal{E}_1 \text{ algebras in chain cplxs}} \overset{\infty\text{-categories}}{\simeq} \underbrace{\mathrm{Alg}_{\mathrm{Ass}}(\mathrm{Ch}_k^{\otimes})}_{\text{dg associative algebras}/k}$$

Not too crazy:

$$A \longleftarrow (A, \ast)$$

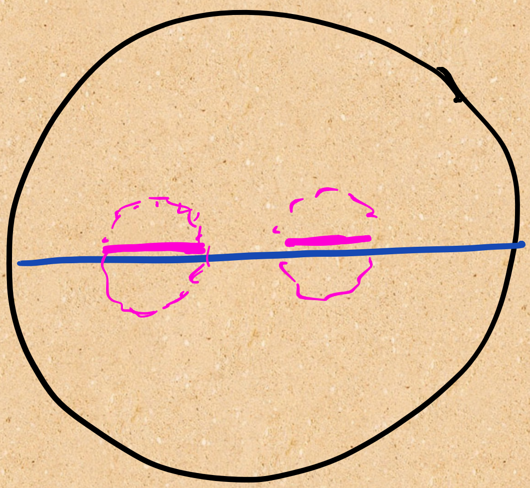



$$A \otimes A \otimes A \xrightarrow{\text{multiplication}} A$$

---

You can "forget" from an $\mathcal{E}_n$-alg to an $\mathcal{E}_{n-1}$-algebra

Ex



$$A \otimes A \to A$$

$$\mathrm{Alg}_{\mathcal{E}_2} \longrightarrow \mathrm{Alg}_{\mathcal{E}_1}$$

$\rightsquigarrow$ Hierarchy of "more commutative" algebras

| 1d | 2d | 3d | | |
|---|---|---|---|---|
| $\mathrm{Alg}_{\mathcal{E}_1} \longleftarrow$ | $\mathrm{Alg}_{\mathcal{E}_2} \longleftarrow$ | $\mathrm{Alg}_{\mathcal{E}_3} \longleftarrow$ | $\cdots \longleftarrow$ | $\mathrm{Alg}_{\mathcal{E}_\infty}$ |
| $\cong$ | | | | $\cong$ |
| $\mathrm{Alg}_{\mathrm{Ass}}$ | | | | $\mathrm{Alg}_{\mathrm{Com}}$ |
| associative | | | | commutative |

What about <u>enveloping</u> algebras?

There's a beautiful generalization:

<u>Thm</u> (Knudsen) $\mathbb{U}_n$ ← enveloping $\mathcal{E}_n$ algebra

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Ch}_k^{\otimes}) \arrow[r, bend left, "\mathbb{U}_n"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathcal{E}_n}(\mathrm{Ch}_k^{\otimes}) \arrow[l, "\text{``forget''}"] \end{tikzcd}$$

where

- $\mathbb{U}_1\mathfrak{g} \simeq U\mathfrak{g}$ as $\mathcal{E}_1$-algs

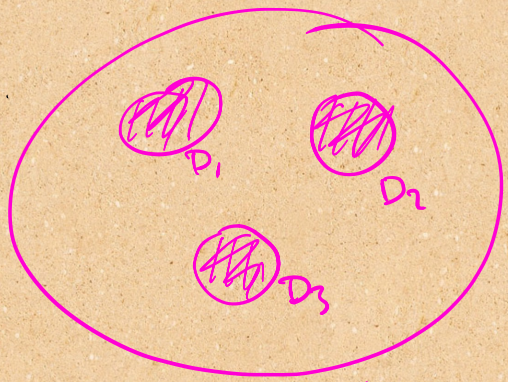
- there is convenient, explicit model

$$\mathbb{U}_n\mathfrak{g}(V) \simeq C_*^{\mathrm{Lie}}\big(\underbrace{\Omega_c^*(V) \otimes \mathfrak{g}}_{\text{dg Lie algebra}}\big)$$

V open set in $\mathbb{R}^n$

Chevalley-Eilenberg chains

compactly supp. de Rham cplx



$D_{big}$

$\mathcal{E}_2$ mult

$$C^{Lie}_{*}\left(\Omega^*_c(D_1 \sqcup D_2 \sqcup D_3) \otimes \mathfrak{g}\right)$$
$$\downarrow$$
$$C^{Lie}_{*}\left(\Omega^*_c(D_{big}) \otimes \mathfrak{g}\right)$$

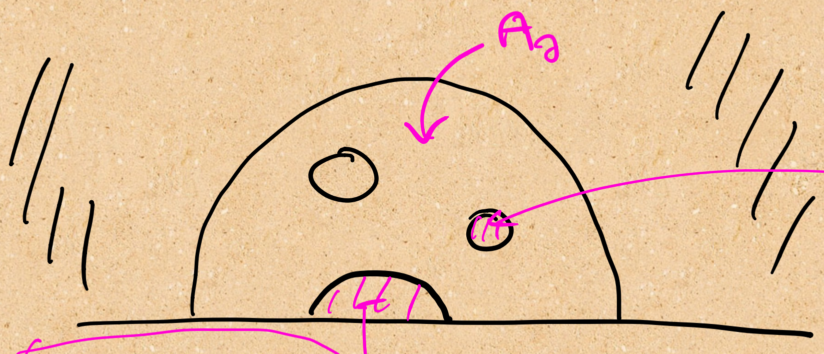
Rmk Knudsen used this to examine config. spaces of mflds but this thing shows up to encode symmetries of n-dim QFTs:

$$\underline{U_n \mathfrak{g}} \xrightarrow{J} \underline{\mathrm{Obs}}^q$$
$$\updownarrow$$
$$\mathfrak{g} \circlearrowright \text{n-dim QFT}$$

symmetries

This is explored in Vol 2 w/ Costello. Rabinovich showed how the axial anomaly & local index fit into this approach.

But let's get back to bulk-bdry systems.



$A_\partial$

$A_{bulk}$ is $\mathcal{E}_2$-algebra

$$A_{bulk} \otimes A_{bulk} \otimes A_\partial \to A_\partial$$

$A_\partial$ is $\mathcal{E}_1$ algebra

Observables of bulk-bdry systems on $\mathbb{H}^{n+1}$ consist of pairs

$$\begin{array}{ccccc} ( A_{bulk} & \circlearrowright & A_\partial ) & \in & Alg_{SC_n} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ Alg_{\mathcal{E}_{n+1}} & & Alg_{\mathcal{E}_n} & & \end{array}$$

swiss cheese operad

$\rightsquigarrow$ configurations of disks & half-disks in $\mathbb{H}^{n+1}$

__Def__ The __$\mathcal{E}_n$-center__ $\mathfrak{Z}_n(A)$ of an $\mathcal{E}_n$-algebra $A$ is the universal $\mathcal{E}_{n+1}$-algebra that fits as part of a $SC_n$-algebra

$$(\mathfrak{Z}_n(A), A).$$

__Thm__ (GGWZ)

For a dg Lie algebra $\mathfrak{g}$, the $\mathcal{E}_n$-center of $\mathbb{U}_n\mathfrak{g}$ is modeled by

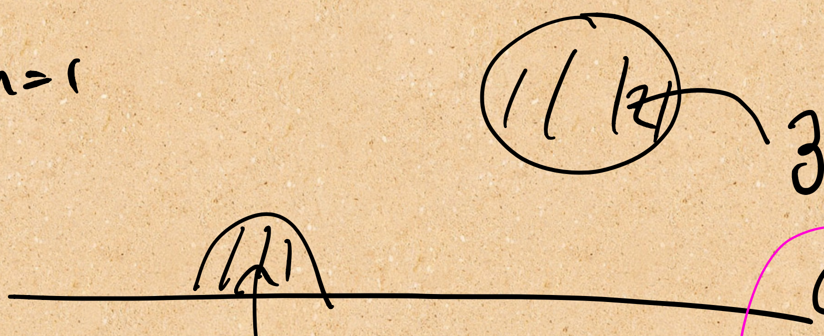
open set in $\mathbb{R}^n$

$$\mathfrak{Z}_n(\mathbb{U}_n\mathfrak{g})(V) \simeq C^*_{Lie}(\mathfrak{g}, \mathbb{U}_n\mathfrak{g})$$

$$\simeq C^*_{Lie}(\underline{\Omega}^*\otimes\mathfrak{g}, C^{Lie}_*(\underline{\Omega}^*_c\otimes\mathfrak{g}))$$

convenient, explicit model

**Ex** $n=1$



$\mathfrak{Z}_1(\mathbb{D}_1\mathfrak{g})$

$\cong$

$C^*(\mathfrak{g}, U\mathfrak{g})$

$\cong$

$\mathrm{Hoch}^*(U\mathfrak{g}, U\mathfrak{g})$

$\mathbb{D}_1\mathfrak{g} \simeq U\mathfrak{g}$

**Consequences**

1. Top. BF theory has a <u>distinguished</u> quantization satisfying $\mathrm{Obs}^q \simeq \mathfrak{Z}_n(\mathbb{D}_n\mathfrak{g})$

2. Lots of fun games looking for generalizations of classic results about $U\mathfrak{g}$

3. $$\int_{M^n} \mathfrak{Z}_n(\mathbb{D}_n\mathfrak{g}) \simeq C^*_{Lie}\left(\Omega^*(M)\otimes\mathfrak{g},\, C^{Lie}_*(\Omega^*_c(M)\otimes\mathfrak{g})\right)$$