Exercise 1. Suppose that $\square ABCD$ is a trapezoid whose sides $AB$ and $CD$ are parallel. Let $S$ denote the intersection of the diagonals of $\square ABCD$. We denote by $A_1, A_2, A_3, A_4$ the areas of the triangles $\triangle ABS, \triangle BCS, \triangle CDS, \triangle DAS$ respectively.

a) Prove that: $A_2 = A_4$.
b) Moreover, prove that: $A_1 \cdot A_3 = A_2^2$.
c) Let $A$ denote the area of $\square ABCD$. Prove that: $A \geq 4A_2$.
d) What can one say about $\square ABCD$ if $A = 4A_2$? Prove your claim.

Solution:

a) Since $AB$ is parallel to $CD$, it follows that the distance from the point $C$ to the line $AB$ equals the distance from the point $D$ to the line $AB$. Hence, the areas of the triangles $\triangle ABC$ and $\triangle ABD$ are equal. The area of the triangle $\triangle ABC$ equals $A_1 + A_2$ and the area of the triangle $\triangle ABD$ equals $A_1 + A_4$. Hence $A_1 + A_2 = A_1 + A_4$, from where we deduce that $A_2 = A_4$.

b) Let $\phi := \angle ASB$. Then, we know that:

$$A_1 = \frac{1}{2} \cdot |AS| \cdot |BS| \cdot \sin \phi$$
$$A_2 = \frac{1}{2} \cdot |BS| \cdot |CS| \cdot \sin(180^\circ - \phi) = \frac{1}{2} \cdot |BS| \cdot |CS| \cdot \sin \phi$$
$$A_3 = \frac{1}{2} \cdot |CS| \cdot |DS| \cdot \sin \phi$$
$$A_4 = \frac{1}{2} \cdot |DS| \cdot |AS| \cdot \sin(180^\circ - \phi) = \frac{1}{2} \cdot |DS| \cdot |AS| \cdot \sin \phi.$$

It follows that:

$$A_1 \cdot A_3 = A_2 \cdot A_4 = \frac{1}{4} \cdot |AS| \cdot |BS| \cdot |CS| \cdot |DS| \cdot \sin^2 \phi.$$

Since $A_2 = A_4$ by part a), it follows that:

$$A_1 \cdot A_3 = A_2^2.$$

c) We know that:

$$A = A_1 + A_2 + A_3 + A_4 = A_1 + A_3 + 2A_2.$$

By the Arithmetic Mean - Geometric Mean Inequality, it follows that:

$$A_1 + A_3 \geq 2\sqrt{A_1 \cdot A_3} = 2A_2.$$

Hence, it follows that:

$$A \geq 2A_2 + 2A_2 = 4A_2.$$

d) From part c), it follows that $A = 4A_2$ if and only if $A_1 + A_3 = 2\sqrt{A_1 \cdot A_3}$. We know that $A_1 + A_3 - 2\sqrt{A_1 \cdot A_3} = (\sqrt{A_1} - \sqrt{A_3})^2$, so equality holds if and only if $A_1 = A_3$. Since $A_1 \cdot A_3 = A_2^2$, it follows that $A = 4A_2$ if and only if $A_1 = A_2 = A_3 = A_4$, which holds if and only if $S$ is the midpoint of $AC$ and of $BD$. The latter is the case if and only if $\square ABCD$ is a parallelogram. $\square$

Exercise 2. Suppose that $A$ and $B$ are distinct $n \times n$ matrices such that:

i) $A^3 = B^3$
Exercise 3. a) Prove that for all $n \in \mathbb{N}$ the following identity holds:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$ 

b) Suppose that $n \in \mathbb{N}$ is a positive integer and suppose that $a_1, a_2, \ldots, a_n$ are mutually distinct positive integers. Prove that:

$$\left(\sum_{j=1}^{n} a_j^5\right) + \left(\sum_{j=1}^{n} a_j^7\right) \geq 2\left(\sum_{j=1}^{n} a_j^3\right)^2$$

c) When does equality hold in part b)?

Solution:

a) We argue by induction on $n$. The base case $n = 1$ holds since both the left and right hand side are equal to 1. For the inductive step, we assume that the claim holds for $n = k$ and we want to show that it holds for $n = k + 1$. This follows from the identity:

$$\left(\frac{(k+1)(k+2)}{2}\right)^2 - \left(\frac{k(k+1)}{2}\right)^2 = (k+1)^2 \cdot 1 \cdot (k+2)^2 - k^2 = (k+1)^2 \cdot \frac{4k+4}{4} = (k+1)^3.$$ 

b) We prove the claim by induction on $n$.

Base case: $n = 1$. Let $x := a_1$. We need to show that:

$$x^5 + x^7 \geq 2(x^3)^2.$$ 

This bound follows from the Arithmetic Mean - Geometric Mean Inequality:

$$x^5 + x^7 \geq 2\sqrt{x^5 \cdot x^7} = 2x^6 = 2(x^3)^2.$$ 

Here, equality holds if and only if $x = 1$.

Inductive step: We suppose that the claim holds for some $n = k \in \mathbb{N}$. We want to show that it holds for $n = k + 1$.

Suppose that $a_1, \ldots, a_k, a_{k+1}$ are mutually distinct positive integers. Let us assume, without loss of generality that $x = a_{k+1}$ is the largest element of the set $\{a_1, \ldots, a_k, a_{k+1}\}$. We then obtain:

$$2\left(\sum_{j=1}^{k+1} a_j^3\right)^2 = 2\left(\sum_{j=1}^{k} a_j^3 + x^3\right)^2 = 2\left(\sum_{j=1}^{k} a_j^3\right)^2 + 4x^3\left(\sum_{j=1}^{k} a_j^3\right) + 2x^6.$$ 

By the inductive assumption, this quantity is:

$$\leq \left(\sum_{j=1}^{k} a_j^3\right) + \left(\sum_{j=1}^{k} a_j^7\right) + 4x^3\left(\sum_{j=1}^{k} a_j^3\right) + 2x^6.$$ 

Since by assumption, $a_1, a_2, \ldots, a_k \leq x - 1 = a_{k+1} - 1$, we obtain that this sum is:

$$\leq \left(\sum_{j=1}^{k} a_j^3\right) + \left(\sum_{j=1}^{k} a_j^7\right) + 4x^3\left(\sum_{j=1}^{k} a_j^3\right) + 2x^6.$$
We now use part a) to deduce that this equals:

\[
\left( \sum_{j=1}^{k} a_j^5 \right) + \left( \sum_{j=1}^{k} a_j^7 \right) + 4x^3 \cdot \left( \frac{(x-1) \cdot x}{2} \right)^2 + 2x^6 =
\]

\[
= \left( \sum_{j=1}^{k} a_j^5 \right) + \left( \sum_{j=1}^{k} a_j^7 \right) + x^7 - 2x^6 + x^5 + 2x^6 =
\]

\[
= \left( \sum_{j=1}^{k} a_j^5 \right) + \left( \sum_{j=1}^{k} a_j^7 \right) + x^7 + x^5 = \left( \sum_{j=1}^{k+1} a_j^5 \right) + \left( \sum_{j=1}^{k+1} a_j^7 \right)
\]

since \( x = a_{k+1} \). The claim now follows.

c) From the proof in part b), it follows that equality holds if and only if \( \{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\} \). The crucial point was that we had equality in \( \sum_{j=1}^{k} a_j^3 = \sum_{j=1}^{x-1} j^3 \). \( \square \)

**Exercise 4.** Let the sequence \((a_n)_{n \geq 0}\) be defined as follows:

i) \( a_0 := 0, a_1 := 1 \).

ii) Given \( a_0, a_1, a_2, \ldots, a_n \), the term \( a_{n+1} \) is defined to be the smallest non-negative integer such that there don’t exist \( i, j \in \{0, 1, \ldots, n\} \), with \( i \leq j \) such that \( a_i, a_j, a_{n+1} \) are three consecutive terms of an arithmetic sequence, i.e. \( a_i + a_{n+1} = 2a_j \).

a) Find \( a_2, a_3 \) and \( a_4 \).
b) Prove that, for \( n \geq 1 \), \( a_n \) equals the \( n \)-th positive integer whose expansion in base 3 doesn’t contain the digit 2.
c) Find \( a_{100} \).

**Solution:**

a) We note that \( a_2 = 3 \) (it can’t equal 2 because \( a_0 = 0, a_1 = 1 \)). Furthermore \( a_3 = 4 \) and \( a_4 = 9 \). We note that \( a_4 \) can’t equal 5 since \( a_1 = 1, a_2 = 3 \). It can’t equal 6 since \( a_0 = 0, a_3 = 3 \). It can’t equal 7 since \( a_1 = 1, a_3 = 4 \). Finally, it can’t equal 8 since \( a_0 = 0, a_3 = 4 \). If we choose \( a_4 = 9 \), then the condition ii) will be satisfied.

b) We argue by induction. Namely, we show that, for all \( k \geq 1 \), \( a_1, \ldots, a_k \) are the first \( k \) positive integers whose expansion in basis 3 doesn’t contain the digit 2.

**Base case:** \( k = 1 \). The claim holds by condition i).

**Inductive step:** Suppose that the claim holds holds for some \( k \geq 1 \). We want to show that it holds for \( k+1 \).

We are given \( a_0, a_1, \ldots, a_k \) and we want to add \( a_{k+1} \) according to the rule ii). Let \( x \) denote the smallest positive integer greater than \( a_k \) which doesn’t contain any digits of 2 in its base three expansion. We want to argue that \( a_{k+1} = x \).

Let us first show that \( a_{k+1} \leq x \). This will follow if we show that for all \( 0 \leq i \leq j \leq k \), the numbers \( a_i, a_j, x \) are not the consecutive terms of an arithmetic sequence, i.e. it is not the case that \( a_i + x = 2a_j \). Suppose that it were the case that \( a_i + x = 2a_j \) for some \( 0 \leq i \leq j \leq k \). Then, we note that the base three expansion of \( 2a_j \) contains only the digits 0 and 2. On the other hand, since \( x \) is strictly bigger than \( a_i \), it follows that there exists a digit where \( x \) has a 1 and where \( a_i \) has a 0. Let’s assume that this is the \( m \)-th digit. In particular, since \( x \) and \( a_i \) only have digits 0 and 1 in base 3, it follows that there are no carries when we add them up and so the \( m \)-th digit of \( x + a_i \) must equal 1. This is a contradiction. Hence, it follows that \( a_{k+1} \leq x \).

We now show that \( a_{k+1} \geq x \). We again argue by contradiction. Suppose that it were the case that \( a_{k+1} < x \). Since \( a_{k+1} > a_k \) (otherwise, we could take \( i = j = k \)), it follows that we would then obtain: \( a_k < a_{k+1} < x \). By construction of \( x \) and by the inductive assumption, it follows that every positive integer which is strictly between \( a_k \) and \( x \) must contain a digit 2 in its base 3 expansion.
Let $y$ and $z$ denote the results of replacing every digit 2 in the base 3 expansion of $a_{k+1}$ by a 0 and by a 1 respectively. Since $a_{k+1}$ was assumed to contain a digit 2 in its base 2 expansion, it follows that:

$$y < z < a_{k+1}$$

and

$$y + a_{k+1} = 2z.$$

Now, $y, z$ contain no digits 2 in their base 3 expansion by definition. Hence, by the inductive assumption, we can find $0 \leq i \leq j \leq k$ such that $y = a_i, z = a_j$. Consequently:

$$a_i + a_{k+1} = 2a_j.$$

This gives us a contradiction.

Hence, it follows that $a_{k+1} \geq x$. Combining this with the fact that $a_{k+1} \leq x$, we now obtain:

$$a_{k+1} = x.$$  

The claim now follows by induction.

c) From part b), we can deduce that, for $n \geq 1$, $a_n$ equals the $n$-th positive integer whose base 3 expansion doesn’t contain the digit 2. In particular, $a_n$ is the result of taking the base 2 expansion of $n$ and replacing the basis of the number system from 2 to 3, e.g. if $n = (1010)_2$ in binary, then $a_n = (1010)_3$, in base 3. In particular, we note that $100 = 64 + 32 + 4 = 2^6 + 2^5 + 2^2 = (1100100)_2$. Hence: $a_{100} = (1100100)_3 = 3^6 + 3^5 + 3^2 = 729 + 243 + 9 = 981. \square$