MATH 114 Sample Midterm 1 Solution

February 7, 2011

Problem 1

Solution:

L_1 and L_2 pass through the same point $(3, -4, 4)$. L_1 is parrallel to vector $\vec{v}_1 = (-1, 1, 2)$ and L_2 is parrallel to vector $\vec{v}_2 = (1, 1, -1)$. So the normal vector is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 \quad = \quad (-3, 1, -2).$$

So the equation of the plane is

$$-3x + y - 2z + d = 0$$

for some d. Plug in the coordinates $(3, -4, 4)$,

$$-9 - 4 - 8 + d = 0 \Rightarrow d = 21.$$

The equation of the plane is $-3x + y - 2z + 21 = 0$.

Problem 2

Solution:

Since $y = f(x) = x^2$, we have $f'(x) = 2x$ and $f''(x) = 2$. So the curvature is

$$\kappa(x) = \frac{f''(x)}{(1 + f'^2(x))^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}},$$

which attains its maximum at $x = 0$, with $\kappa_{\max} = 2$. So the osculating circle has minimum radius at $x = 0$ with $\rho_{\min} = 1/\kappa_{\max} = 1/2$. The equation of the circle is

$$x^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2.$$

Problem 3

Solution:

The velocity vector is

$$\frac{d\vec{r}}{dt} = (\cos t, -2, -\sin t)$$

So the arclength is

$$\begin{aligned} l &= \int_{\pi}^{2\pi} \sqrt{(\cos t)^2 + (-2)^2 + (\sin t)^2} dt \\ &= \int_{\pi}^{2\pi} \sqrt{5} dt \\ &= \sqrt{5}\pi \end{aligned}$$

Problem 4

Solution:
If $\vec{u}$ and $\vec{v}$ are orthogonal, we have

$$0 = \vec{u} \cdot \vec{v} = k^2 - 6 - k$$

The solutions are $k = -2$ or $k = 3$.

Problem 5

Solution:
The function F is

$$F(r, l) = (V(r, l), S(r, l)) = (\pi r^2 l, 2\pi r l + 2\pi r^2)$$

with derivative

$$\begin{bmatrix} \frac{\partial V}{\partial r} & \frac{\partial V}{\partial l} \\ \frac{\partial S}{\partial r} & \frac{\partial S}{\partial l} \end{bmatrix} = \begin{bmatrix} 2\pi r l & \pi r^2 \\ 2\pi l + 4\pi r & 2\pi r \end{bmatrix}$$

Problem 6

Solution:
Let $\vec{v}_1 = (1, 2, 0)$ and $\vec{v}_2 = (3, 0, 2)$. Then the area of the parallelogram spanned by them is

$$S = |\vec{v}_1 \times \vec{v}_2| = |(4, -2, -6)| = \sqrt{(4)^2 + (-2)^2 + (-6^2)} = 2\sqrt{14}.$$

Problem 7

Solution:

$$\frac{d\vec{r}(t)}{dt} = (\omega \sinh(\omega t), \omega \cosh(\omega t)).$$

$$\frac{d^2\vec{r}(t)}{dt^2} = (\omega^2 \cosh(\omega t), \omega^2 \sinh(\omega t)) = \omega^2 \vec{r}(t).$$

So the scalar factor is ω^2.

Problem 8

Solution:
All pairs of points have distance $\sqrt{2}$ and the angle between any two intersectiong edges is $\pi/3$.

Problem 9

Solution:
Denote the function for volume, area and cost as V, A, C. It is easy to show that $V(h, w, d) = whd$, $A(h, w, d) = 2wh + 2hd + 2wd$ and $C(h, w, d) = 13wh + 6hd + 6wd$. Their derivatives are

$$V' = (wd, hd, wh) = (16, 16, 16),$$

$$A' = (2w + 2d, 2h + 2d, 2h + 2w) = (16, 16, 16),$$
$$C' = (13w + 6d, 13h + 6d, 6h + 6w) = (76, 76, 48).$$

The dimension with least cost increase is the depth. Because $\partial C/\partial d$ is the smallest among three dimensions.
If $dh/dt = 2$, $dw/dt = 3$ and $dd/dt = 0$, then we have

$$\frac{dV}{dt} = \frac{\partial V}{\partial h}\frac{dh}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial d}\frac{dd}{dt} = 16 \cdot 2 + 16 \cdot 3 + 16 \cdot 0 = 80,$$
$$\frac{dA}{dt} = \frac{\partial A}{\partial h}\frac{dh}{dt} + \frac{\partial A}{\partial w}\frac{dw}{dt} + \frac{\partial A}{\partial d}\frac{dd}{dt} = 16 \cdot 2 + 16 \cdot 3 + 16 \cdot 0 = 80,$$
$$\frac{dC}{dt} = \frac{\partial C}{\partial h}\frac{dh}{dt} + \frac{\partial C}{\partial w}\frac{dw}{dt} + \frac{\partial C}{\partial d}\frac{dd}{dt} = 76 \cdot 2 + 76 \cdot 3 + 48 \cdot 0 = 380.$$

Problem 10

Solution:
(To make typing easier, I have changed the n-by-1 vector to 1-by-n. Don't do this when you are taking the exam) The two particles intersect when $\vec{r}(t) = \vec{s}(t)$ for $t > 0$. We have

$$\vec{r}(t) - \vec{s}(t) = (5t - t^2, 5t - t^2, 5t - t^2)$$

So the two particles intersect when $t = 5$.
For particle $\vec{r}$, the unit tangent vector is

$$\mathbf{T}_r(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{(1, 2, -1)}{\sqrt{6}} = (\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$$

Since $\mathbf{T}_r'(t) = 0$, the normal vector is undefined. But the curvature is $\kappa_r(5) = 0$, and the unit tangent vector is $\mathbf{T}_r(5) = (\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$.
For particle $\vec{s}$, the unit tangent vector is

$$\mathbf{T}_s(t) = \frac{\vec{s}'(t)}{|\vec{s}'(t)|} = \frac{(-4 + 2t, 7 - 2t, -6 + 2t)}{\sqrt{12t^2 - 68t + 101}} = f(t)(-4 + 2t, 7 - 2t, -6 + 2t)$$

where

$$f(t) = (12t^2 - 68t + 101)^{-\frac{1}{2}} = (12t^2 - 68t + 101)^{-\frac{3}{2}}(12t^2 - 68t + 101)$$

$$f'(t) = -\frac{1}{2}(12t^2 - 68t + 101)^{-\frac{3}{2}}(24t - 68) = (12t^2 - 68t + 101)^{-\frac{3}{2}}(-12t + 34)$$

So the unit normal vector is

$$\mathbf{N}_s(t) = \frac{\mathbf{T}_s'(t)}{|\mathbf{T}_s'(t)|} = \frac{(48t + 66, -84t + 36, 72t - 2)}{\sqrt{(48t + 66)^2 + (-84t + 36)^2 + (72t - 2)^2}}$$

Plug in $t = 5$, we have

$$\mathbf{T}_s(5) = \frac{(6, -3, 4)}{\sqrt{61}} = (\frac{6}{\sqrt{61}}, \frac{-3}{\sqrt{61}}, \frac{4}{\sqrt{61}}),$$

$$\mathbf{N}_s(5) = \frac{(306, -384, 358)}{\sqrt{369256}} = (\frac{153}{\sqrt{92314}}, -\frac{192}{\sqrt{92314}}, \frac{179}{\sqrt{92314}})$$

For the curvature κ_s, we have to compute the velocity and acceleration

$$\vec{s}'(t) = (-4 + 2t, 7 - 2t, -6 + 2t), \ \vec{s}''(t) = (2, -2, 2)$$

so the curvature is

$$\kappa_s(5) = \frac{|(6, -3, 4) \times (2, -2, 2)|}{|(6, -3, 4)|^3} = \frac{2\sqrt{14}}{61\sqrt{61}}.$$

Problem 11

Solution:

The function is $S(\rho, \theta, \phi) = (\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$. The determinant of the derivative matrix is

$$\begin{aligned} DS &= \begin{vmatrix} \cos\theta \sin\phi & -\rho \sin\theta \sin\phi & \rho \cos\theta \cos\phi \\ \sin\theta \sin\phi & \rho \cos\theta \sin\phi & \rho \sin\theta \cos\phi \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix} \\ &= \rho^2(-\cos^2\theta \sin^3\phi - \sin^2\theta \sin\phi \cos^2\phi - \cos^2\theta \sin\phi \cos^2\phi - \sin^2\theta \sin^3\phi) \\ &= \rho^2(-\sin^3\phi - \sin\phi \cos^2\phi) \\ &= -\rho^2 \sin\phi \end{aligned}$$

Problem 12

Solution:

Since $PV = nRT$, then

$$P = \frac{nRT}{V}, \ V = \frac{nRT}{P}, \ T = \frac{PV}{nR}.$$

Then the partial derivative is

$$\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}, \ \frac{\partial V}{\partial T} = \frac{nR}{P}, \ \frac{\partial T}{\partial P} = \frac{V}{nR}.$$

So their product is

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -\frac{n^2R^2TV}{V^2PnR} = -\frac{nRT}{PV} = -1.$$