## 114 SPRING 2011: Calculus MIDTERM I (WITH SOLUTIONS)

PROBLEM 1: (10 pts) A particle is at the origin at $t=0$; it accelerates according to

$$
\vec{a}(t)=2 \hat{\imath}+8 t \hat{\jmath}+12 t^{2} \hat{k}
$$

with initial velocity $\vec{v}(0)=\hat{\jmath}+2 \hat{k}$. What is the position of the particle at time $t=3$ ?
(A) $6 \hat{\imath}+37 \hat{\jmath}+110 \hat{k}$
(B) $9 \hat{\imath}+36 \hat{\jmath}+33 \hat{k}$
(C) $9 \hat{\imath}+36 \hat{\jmath}+85 \hat{k}$
(D) $9 \hat{\imath}+37 \hat{\jmath}+83 \hat{k}$
(E) $6 \hat{\imath}+108 \hat{\jmath}+83 \hat{k}$
(F) $9 \hat{\imath}+39 \hat{\jmath}+87 \hat{k}$

## SOLUTION:

We have the differential equation $\vec{v}^{\prime}(t)=\vec{a}(t)$, whose solution is

$$
\begin{aligned}
\vec{v}(t) & =\vec{v}(0)+\int_{0}^{t} \vec{a}(s) \mathrm{d} s \\
& =(\hat{\jmath}+2 \hat{k})+\left(2 t \hat{\imath}+4 t^{2} \hat{\jmath}+4 t^{3} \hat{k}\right) \\
& =2 t \hat{\imath}+\left(4 t^{2}+1\right) \hat{\jmath}+\left(4 t^{3}+2\right) \hat{k}
\end{aligned}
$$

Also $\vec{r}^{\prime}(t)=\vec{v}(t)$, so

$$
\begin{aligned}
\vec{r}(t) & =\vec{r}(0)+\int_{0}^{t} \vec{v}(s) \mathrm{d} s \\
& =t^{2} \hat{\imath}+\left(\frac{4}{3} t^{3}+t\right) \hat{\jmath}+\left(t^{4}+2 t\right) \hat{k}
\end{aligned}
$$

Evaluating at $t=3$, we get

$$
\vec{r}(3)=9 \hat{\imath}+39 \hat{\jmath}+87 \hat{k}
$$

PROBLEM 2: (8 pts) What is the volume of the parallelepiped spanned by the vectors

$$
\vec{u}=\left(\begin{array}{l}
1 \\
2 \\
4
\end{array}\right) \quad \vec{v}=\left(\begin{array}{c}
-1 \\
2 \\
0
\end{array}\right) \quad \vec{w}=\left(\begin{array}{l}
3 \\
0 \\
1
\end{array}\right)
$$

Hint: use a determinant!
(A) -24
(B) 24
(C) -22
(D) 22
(E) -20
(F) 20

## SOLUTION:

The volume of the parallelepiped spanned by the vectors spanned by three vectors is given by the magnitude of their scalar triple product:

$$
\begin{aligned}
\vec{u} \cdot(\vec{v} \times \vec{w}) & =\left|\begin{array}{ccc}
1 & 2 & 4 \\
-1 & 2 & 0 \\
3 & 0 & 1
\end{array}\right|=1\left|\begin{array}{ll}
2 & 0 \\
0 & 1
\end{array}\right|-2\left|\begin{array}{cc}
-1 & 0 \\
3 & 1
\end{array}\right|+4\left|\begin{array}{cc}
-1 & 2 \\
3 & 0
\end{array}\right| \\
& =2+2-24=-20
\end{aligned}
$$

and hence $V=|\vec{u} \cdot(\vec{v} \times \vec{w})|=20$.

PROBLEM 3: (9 pts) For which values of constants $a$ and $b$ are the following two planes parallel?

$$
a x+8 y+b z=-1 \quad ; \quad b x+a y+z=5
$$

(A) There are no values of $a$ and $b$ making the planes parallel.
(B) $a=2$ and $b=4$
(C) $a=-1$ and $b=3$
(D) $a=-1$ and $b=5$
(E) $a=4$ and $b=2$
(F) $a=8$ and $b=8$

## SOLUTION:

Two planes are parallel if their normal vectors are, and two nonzero vectors are parallel whenever their cross product is zero. A normal vector to the first plane is $\vec{n}_{1}=a \hat{\imath}+8 \hat{\jmath}+b \hat{k}$; for the second plane, we can choose $\vec{n}_{2}=b \hat{\imath}+a \hat{\jmath}+\hat{k}$. Now,

$$
\vec{n}_{1} \times \vec{n}_{2}=\left|\begin{array}{ccc}
\hat{\imath} & \hat{\jmath} & \hat{k} \\
a & 8 & b \\
b & a & 1
\end{array}\right|=(8-a b) \hat{\imath}+\left(b^{2}-a\right) \hat{\jmath}+\left(a^{2}-8 b\right) \hat{k}
$$

Hence the two planes are parallel if

$$
\begin{aligned}
8-a b & =0 \\
b^{2}-a & =0 \\
a^{2}-8 b & =0
\end{aligned}
$$

From the second equation, we get $a=b^{2}$. Substituting into the first equation, we arrive at $b^{3}=8$, i.e., $b=2$, which implies $a=4$. It is immediate to verify that these values of $a$ and $b$ also satisfy the third equation.

PROBLEM 4: (10 pts) A child pulls back a pellet in a slingshot and fires. The function $S$ that determines $d$, the distance the pellet travels and $h$ the maximal height of the pellet, as a function of $\ell$ the distance the rubber band is pulled back, and $\theta$ the angle of inclination from the ground, is

$$
S\binom{\ell}{\theta}=\binom{d}{h}=\binom{100 \ell \sin (2 \theta)}{10 \ell^{2} \sin ^{2}(\theta)}
$$

Assume that the pull-distance, $\ell$, equals 2 and the angle, $\theta$, is $\pi / 4$ (to achieve optimal distance). Assume that these inputs are changing at the rates $\dot{\ell}=1$ and $\dot{\theta}=3$. At what rates do the outputs $d$ and $h$ change?
(A) $\dot{d}=100$ and $\dot{h}=140$
(B) $\dot{d}=300$ and $\dot{h}=100$
(C) $\dot{d}=160$ and $\dot{h}=120$
(D) $\dot{d}=200$ and $\dot{h}=100$
(E) $\dot{d}=100$ and $\dot{h}=280$
(F) $\dot{d}=1300$ and $\dot{h}=140$

## SOLUTION:

The total rate of change of $d \mathrm{~s}$ given by

$$
\begin{aligned}
\dot{d} & =\left.d_{\ell}\right|_{(2, \pi / 4)} \dot{\ell}+\left.d_{\theta}\right|_{(2, \pi / 4)} \dot{\theta} \\
& =\left.(100 \sin (2 \theta))\right|_{(2, \pi / 4)} \dot{\ell}+\left.(200 \ell \cos (2 \theta))\right|_{(2, \pi / 4)} \dot{\theta} \\
& =100 \sin (\pi / 2) \cdot 1+200 \cdot 2 \cos (\pi / 2) \cdot 3=100
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\dot{h} & =\left.h_{\ell}\right|_{(2, \pi / 4)} \dot{\ell}+\left.h_{\theta}\right|_{(2, \pi / 4)} \dot{\theta} \\
& =\left.\left(20 \ell \sin ^{2} \theta\right)\right|_{(2, \pi / 4)} \dot{\ell}+\left.\left(20 \ell^{2} \sin \theta \cos \theta\right)\right|_{(2, \pi / 4)} \dot{\theta} \\
& =20 \cdot 2 \sin ^{2}(\pi / 4) \cdot 1+20 \cdot 2^{2} \sin (\pi / 4) \cos (\pi / 4) \cdot 3=140
\end{aligned}
$$

PROBLEM 5: (10 pts) Consider the function

$$
F\left(x_{1}, x_{2}, x_{3}, \ldots, x_{n}\right)=x_{1} x_{2} x_{3} \ldots x_{n-1} x_{n}
$$

that returns the product of the $n$ inputs $x_{1}, \ldots, x_{n}$. At the point where all $x_{i}=2$ and all the $x_{i}$ are increasing at rate 3 , what is the rate of change of the output?
(A) $3 n$
(B) $(3)(2 n)$
(C) $(3)\left(2^{n}\right)$
(D) $(3 n)\left(2^{n-1}\right)$
(E) $\left(3^{n}\right)\left(2^{n-1}\right)$
(F) $\left(3^{n}\right)\left(2^{n}\right)$

## SOLUTION:

First of all, notice that the rate of change of the output with respect to the input $x_{i}$ is the product of the rest of the inputs:

$$
\left.F_{x_{i}}\right|_{(2, \ldots, 2)}=\left.\prod_{\substack{1 \leq j \leq n \\ j \neq i}} x_{j}\right|_{(2, \ldots, 2)}=2^{n-1}
$$

The total rate of change of the output is then given by:

$$
\begin{aligned}
\left.\dot{F}\right|_{(2, \ldots, 2)} & =\left.F_{x_{1}}\right|_{(2, \ldots, 2)} \dot{x}_{1}+\left.F_{x_{2}}\right|_{(2, \ldots, 2)} \dot{x}_{2}+\cdots+\left.F_{x_{n}}\right|_{(2, \ldots, 2)} \dot{x}_{n} \\
& =2^{n-1} \cdot 3+2^{n-1} \cdot 3+\cdots+2^{n-1} \cdot 3 \\
& =(3 n)\left(2^{n-1}\right)
\end{aligned}
$$

since there are $n$ terms in the sum.

PROBLEM 6: (8 pts) Recall that the curvature $\kappa(t)$ of a curve $\vec{r}(t)$ is given by

$$
\kappa(t)=\frac{\left\|\frac{d \vec{T}(t)}{d t}\right\|}{\left\|\frac{d \vec{r}(t)}{d t}\right\|}
$$

where $\vec{T}(t)$ is the unit tangent vector. Use this formula to compute the [constant!] curvature of the helix

$$
\vec{r}(t)=\left(\begin{array}{c}
a \cos b t \\
a \sin b t \\
b t
\end{array}\right)
$$

(A) $\frac{a b}{\sqrt{a^{2}+1}}$
(B) $\frac{1}{a}$
(C) $\frac{a}{a^{2}+1}$
(D) $\frac{1}{a b}$
(E) $\frac{1}{b \sqrt{a^{2}+1}}$
(F) $\frac{a}{\sqrt{a^{2}+1}}$

## SOLUTION:

This problem is a straightforward computation:

$$
\begin{gathered}
\vec{r}^{\prime}(t)=\left(\begin{array}{c}
-a b \sin b t \\
a b \cos b t \\
b
\end{array}\right), \quad\left\|\vec{r}^{\prime}(t)\right\|=b \sqrt{a^{2}+1} \\
\vec{T}(t)=\frac{1}{b \sqrt{a^{2}+1}}\left(\begin{array}{c}
-a b \sin b t \\
a b \cos b t \\
b
\end{array}\right) \\
\vec{T}^{\prime}(t)=\frac{1}{b \sqrt{a^{2}+1}}\left(\begin{array}{c}
-a b^{2} \cos b t \\
-a b^{2} \sin b t \\
0
\end{array}\right), \quad\left\|\vec{T}^{\prime}(t)\right\|=\frac{a b}{\sqrt{a^{2}+1}} \\
\kappa(t)=\frac{a b / \sqrt{a^{2}+1}}{b \sqrt{a^{2}+1}}=\frac{a}{a^{2}+1}
\end{gathered}
$$

PROBLEM 7: (8 pts) Which of the following best describes the intersection of the paraboloid

$$
x^{2}+z^{2}=-3 y
$$

with the sphere at the origin of radius 2? Please show me the derivation of your answer: don't simply guess!
(A) Two circles of radius 3 in the $x-z$ planes $y=-1$ and $y=4$.
(B) A circle of radius 3 in the $x-z$ plane $y=-1$.
(C) A parabola of the form $x=y^{2}-3 y-4$.
(D) A circle of radius $\sqrt{3}$ in the $x-z$ plane $y=-1$.
(E) Two circles of radius $\sqrt{3}$ in the $x-z$ planes $y=-1$ and $y=4$.
(F) The intersection is empty.

## SOLUTION:

The equation of the sphere or radius 2 centered at the origin is $x^{2}+y^{2}+z^{2}=4$. Eliminating $x^{2}+z^{2}$ between both equations yields $y^{2}-3 y-4=0$, which is satisfied for $y=-1$ and $y=4$.
If $y=-1$, the equations of both the paraboloid and the sphere reduce to $x^{2}+z^{2}=3$, which describes a circle of radius $\sqrt{3}$.
If $y=4$, the equation of the paraboloid gives $x^{2}+z^{2}=-12$, and this does not have any real solutions.

PROBLEM 8: (9 pts) Here's an interesting problem. The following line and plane intersect at the origin:

$$
\left(\begin{array}{l}
x \\
y \\
z
\end{array}\right)=\left(\begin{array}{l}
4 t \\
7 t \\
4 t
\end{array}\right) \quad ; \quad 6 x+2 y+3 z=0
$$

What is the angle of intersection between the two? No, I am not giving you a formula for that - you will have to figure it out. (1) Draw a cartoon picture of the situation; (2) What is the vector of the line orthogonal to the plane? (3) Can you figure out the angle between the lines orthogonal to and crossing the plane using a formula you know? Of course you can... (4) Use that angle to determine the angle between the line and plane. Explain what you are doing please so I can award partial credit...
(A) 0
(B) $\arctan \frac{50}{63}$
(C) $\arcsin \frac{50}{63}$
(D) $\arccos \frac{50}{63}$
(E) $\frac{\pi}{2}$
(F) $\frac{\pi}{2}-\arccos \frac{50}{63}$

## SOLUTION:

PROBLEM 9: (9 pts) Which (one and only one) of the following vectors is tangent to the plane passing through the points

$$
\left(\begin{array}{l}
1 \\
2 \\
0
\end{array}\right) \quad\left(\begin{array}{c}
2 \\
3 \\
-1
\end{array}\right) \quad\left(\begin{array}{l}
3 \\
5 \\
1
\end{array}\right)
$$

(A) $\hat{\imath}-\hat{\jmath}-\hat{k}$
(B) $-2 \hat{\imath}+\hat{\jmath}-\hat{k}$
(C) $\hat{\imath}+\hat{\jmath}-\hat{k}$
(D) $4 \hat{\imath}-3 \hat{\jmath}+\hat{k}$
(E) $\hat{\imath}-2 \hat{\jmath}+\hat{k}$
(F) $4 \hat{\imath}+3 \hat{\jmath}+\hat{k}$

## SOLUTION:

From the three points in the plane, we can form two vectors that are parallel to the plane, namely the vector that starts at the first point and ends at the second, $\hat{\imath}+\hat{\jmath}-\hat{k}$, and the one that goes from the first point to the third: $2 \hat{\imath}+4 \hat{\jmath}+\hat{k}$. The cross product of these gives a vector $\vec{n}$ normal to the plane:

$$
\vec{n}=(\hat{\imath}+\hat{\jmath}-\hat{k}) \times(2 \hat{\imath}+4 \hat{\jmath}+\hat{k})=\left|\begin{array}{ccc}
\hat{\imath} & \hat{\jmath} & \hat{k} \\
1 & 1 & -1 \\
2 & 4 & 1
\end{array}\right|=5 \hat{\imath}-3 \hat{\jmath}+2 \hat{k}
$$

A nonzero vector is tangent to the plane if and only if it is perpendicular to $\vec{n}$, i.e., if its dot product with $\vec{n}$ is zero. Hence we just need to calculate:
(A) $(\hat{\imath}-\hat{\jmath}-\hat{k}) \cdot \vec{n}=6$
(B) $(-2 \hat{\imath}+\hat{\jmath}-\hat{k}) \cdot \vec{n}=-15$
(C) $(\hat{\imath}+\hat{\jmath}-\hat{k}) \cdot \vec{n}=0$
(D) $(4 \hat{\imath}-3 \hat{\jmath}+\hat{k}) \cdot \vec{n}=31$
(E) $(\hat{\imath}-2 \hat{\jmath}+\hat{k}) \cdot \vec{n}=13$
(F) $(4 \hat{\imath}+3 \hat{\jmath}+\hat{k}) \cdot \vec{n}=13$

PROBLEM 10: (10 pts) Compute the arclength of the curve

$$
\vec{r}(t)=\left(\begin{array}{c}
t \cos t \\
t \sin t \\
\frac{2 \sqrt{2}}{3} t^{\frac{3}{2}}
\end{array}\right)
$$

as $t$ goes from 0 to $2 \pi$.
(A) $4 \pi^{2}+2 \pi$
(B) $\pi^{2}$
(C) $2 \pi$
(D) $4 \pi^{2}+4 \pi+1$
(E) $2 \pi(\pi+1)$
(F) $2 \pi^{2}+1$

## SOLUTION:

The arclength is given by integrating the magnitude of the derivative $\vec{r}^{\prime}(t)$, so we start by calculating the latter:

$$
\vec{r}^{\prime}(t)=\left(\begin{array}{c}
\cos t-t \sin t \\
\sin t+t \cos t \\
\sqrt{2 t}
\end{array}\right)
$$

$$
\begin{aligned}
\left\|\vec{r}^{\prime}(t)\right\| & =\sqrt{(\cos t-t \sin t)^{2}+(\sin t+t \cos t)^{2}+2 t} \\
& =\sqrt{\cos ^{2} t-2 t \sin t \cos t+t^{2} \sin ^{2} t+\sin ^{2} t+2 t \sin t \cos t+t^{2} \cos ^{2} t+2 t} \\
& =\sqrt{1+2 t+t^{2}}=\sqrt{(1+t)^{2}}=1+t
\end{aligned}
$$

The integral is now easy:

$$
s=\int_{0}^{2 \pi}\left\|\vec{r}^{\prime}(t)\right\| \mathrm{d} t=\int_{0}^{2 \pi}(1+t) \mathrm{d} t=t+\left.\frac{t^{2}}{2}\right|_{0} ^{2 \pi}=2 \pi+2 \pi^{2}
$$

PROBLEM 11: (9 pts) Can you compute the length of a vector in $\mathbb{R}^{3}$ ? Of course you can: 3-d is "physical". How about $\mathbb{R}^{\infty}$ ? Consider an experiment that gives you a reading $u(i)$ each second $i=1,2,3, \ldots$. If you store these data in a "time-series" - a vector $\vec{u}$ with an infinite number of slots - how do you compute its length or compare two such vectors? To get you thinking in terms of time series vectors, try this: what is the length of the following vector in $\mathbb{R}^{\infty}$ ?

$$
\vec{u}=\left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots\right)
$$

(Such data might come from a physical process undergoing exponential decay, like radioactive carbon isotopes...)
(A) $\frac{1}{4}$
(B) $\frac{1}{2}$
(C) $2 \sqrt{3}$
(D) $\frac{4}{3}$
(E) $\frac{2}{\sqrt{3}}$
(F) $\sqrt{2}$

## SOLUTION:

The length of a vector in $\mathbb{R}^{3}$ is given by the square root of its dot product with itself. Notice that this is also the case for a vector in $\mathbb{R}^{2}$. We can then try defining the length of a vector in $\mathbb{R}^{\infty}$ in the same way:

$$
\|\vec{u}\|=\sqrt{\vec{u} \cdot \vec{u}}=\sqrt{\sum_{i=1}^{\infty} u_{i}^{2}}
$$

In the particular case above, this definition gives

$$
\|\vec{u}\|=\sqrt{\sum_{i=1}^{\infty}\left(\frac{1}{2^{i-1}}\right)^{2}}=\sqrt{\sum_{i=1}^{\infty} 2^{2-2 i}}=\sqrt{\frac{1}{1-2^{-2}}}=\frac{2}{\sqrt{3}}
$$

(do you remember the formula for the sum of a geometric series?)

