MATH 114 Sample Midterm 2 Solution

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Problem 1

Solution:

Note: (1,2,3) isn't on the surface! So we change the equation to $x^3 - xy^2 + 4xz = 9$

Rearrange the equation as $4xz = 9 - x^3 + xy^2$, so

$$z = f(x,y) = \frac{1}{4}(-x^2 + y^2 + \frac{9}{x}).$$

The first partial derivatives are

$$f_x(x,y) = -\tfrac{x}{2} - \tfrac{9}{4x^2},\ f_x(1,2) = -\tfrac{11}{4}$$
$$f_y(x,y) = \tfrac{y}{2},\ f_y(1,2) = 1$$

Hence the tangent plan at the given point is given by the following equation

$$(z-3) = -\frac{11}{4}(x-1) + (y-2)$$

Thank Rafael Pelles for pointing out a mistake in the old solution.

Problem 2

Solution:

$$\begin{aligned} D_{(0,0,0)}(f \circ g) &= D_{g(0,0,0)}f \cdot D_{(0,0,0)}g \\ &= D_{(0,0,\pi)}f \cdot D_{(0,0,0)}g, \end{aligned}$$

where

$$\begin{aligned} D_{(0,0,\pi)}f &= \left.\begin{bmatrix} -vw\sin(uvw) & -uw\sin(uvw) & -uv\sin(uvw) \\ vw\cos(uvw) & uw\cos(uvw) & uv\cos(uvw) \\ \sec^2(u+v+w) & \sec^2(u+v+w) & \sec^2(u+v+w) \end{bmatrix}\right|_{(0,0,\pi)} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} D_{(0,0,0)}g &= \left[\begin{array}{ccc} 1 & 1 & 1 \\ yz & xz & xy \\ \pi yz e^{xyz} & \pi xz e^{xyz} & \pi xy e^{xyz} \end{array}\right]\Bigg|_{(0,0,0)} \\ &= \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]. \end{aligned}$$

Hence

$$D_{(0,0,0)}(f \circ g) = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{array}\right].$$

Problem 3

Expand the equation of the ellipse at (1,1) as

$$\begin{aligned} 0 &= 2x^2 + y^2 - 3 \\ &= 2[(x-1)+1]^2 + [(y-1)+1]^2 - 3 \\ &= 2(x-1)^2 + 4(x-1) + 2 + (y-1)^2 + 2(y-1) + 1 - 3 \\ &= 4(x-1) + 2(y-1) + o(2). \end{aligned}$$

Drop the non-linear terms (quadratic terms here) so we have the equation of the tangent line

$$0 = 4(x-1) + 2(y-1)$$

Problem 4

We have the derivative of f and the Hessian of f as

$$\begin{aligned} Df(1,-2,1) &= (3x^2 - 2y + 2xz, -2x, x^2 + 7)\Big|_{(1,-2,1)} = (9,-2,8) \\ Hf(1,-2,1) &= \left[\begin{array}{ccc} 6x+2z & -2 & 2x \\ -2 & 0 & 0 \\ 2x & 0 & 0 \end{array}\right]\Bigg|_{(1,-2,1)} = \left[\begin{array}{ccc} 8 & -2 & 2 \\ -2 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right]. \end{aligned}$$

So the taylor expansion up to order 2 is

$$\begin{aligned} f(x,y,z) &\approx f(1,-2,1) + d \cdot Df + \frac{1}{2} d'[Hf]d \\ &= 13 + 9(x-1) - 2(y+2) + 8(z-1) \\ &\quad +4(x-1)^2 - 2(x-1)(y+2) + 2(x-1)(z-1). \end{aligned}$$

x produces the largest change in f when varied because the coefficient for $(x-1)$ has the largest absolute value among all the three (9, -2 and 8).

Problem 5

Note that

$$\nabla f\Big|_{(1,1,0)} = (2x, -2y, -2z)\Big|_{(1,1,0)} = (2, -2, 0).$$

So the direction with deepest decrease is $(-2, 2, 0)$.

Problem 6

Let $f(x, y) = x^3 - 12xy + 8y^3$. Then the derivative and Hessian are

$$\begin{aligned} Df(x,y) &= (3x^2 - 12y, -12x + 24y^2) \\ Hf(x,y) &= \begin{bmatrix} 6x & -12 \\ -12 & 48y \end{bmatrix}. \end{aligned}$$

By setting $Df = 0$, we have two critical points (0,0) and (2,1). For (0,0), $\det[Hf] = -144 < 0$. So (0,0) is a saddle point. For (2,1), $\det[Hf] = 432 > 0$ and $f_{xx} = 12 > 0$, so (2,1) is a local minimum.

Problem 7

Assume the silo has the cylinder part with radius r and height h. Then the volume and the cost functions are

$$\begin{aligned} V(r,h) &= \pi r^2 h + \frac{2}{3}\pi r^3, \\ C(r,h) &= \pi r^2 + 4\pi r h + 10\pi r^2. \end{aligned}$$

With the constraint $V(r, h) = 900\pi$, we want to minimze $C(r, h)$. The Langrange multiplier method can be applied as follows,

$$\begin{aligned} \nabla V &= (2\pi r h + 2\pi r^2, \pi r^2) \\ \nabla C &= (2\pi r + 4\pi h + 20\pi r, 4\pi r) = (22\pi r + 4\pi h, 4\pi r) \end{aligned}$$

The extreme value is attained when $\nabla C = \lambda \nabla V$ for some λ. By comparing the second coordinate we know $\lambda = 4/r$. Then plug that into the equation for first coordinate, we have

$$\begin{aligned} 8\pi h + 8\pi r &= 22\pi r + 4\pi h \\ (8-4)\pi h &= (22-8)\pi r \\ h &= \frac{7}{2} r \end{aligned}$$

Plug this equation into $V(r, h) = 900\pi$, the solution is

$$\begin{aligned} r &= 6 \\ h &= 21 \end{aligned}$$

Thank Spencer Lee for pointing out a mistake in the old solution.

Problem 8

$$
\begin{aligned}
& \sin(y+z+e^{xy}-cos(xz)) \\
= \; & \sin\left[y+z+\left(1+xy+\frac{x^2y^2}{2}+o(4)\right)-\left(1-\frac{x^2z^2}{2}+o(4)\right)\right] \\
= \; & \sin\left[y+z+xy+\frac{x^2y^2}{2}+\frac{x^2z^2}{2}+o(4)\right] \\
= \; & \left[y+z+xy+\frac{x^2y^2}{2}+\frac{x^2z^2}{2}\right]-\frac{1}{3!}\left[y+z+xy+\frac{x^2y^2}{2}+\frac{x^2z^2}{2}\right]^3+o(4) \\
= \; & y+z+xy+\frac{x^2y^2}{2}+\frac{x^2z^2}{2}-\frac{1}{6}\left[(y+z)^3+3xy(y+z)\right]+o(4) \\
\approx \; & y+z+xy-\frac{y^3}{6}-\frac{3y^2z}{6}-\frac{3yz^2}{6}-\frac{z^3}{6}-\frac{3xy^2}{6}-\frac{3xyz}{6}+\frac{x^2y^2}{2}+\frac{x^2z^2}{2}.
\end{aligned}
$$

Note that the second last equation is because we can pick out all the products which won't blow the order limit 4. In this case we can choose the cube of order 1 terms $(y+z)$, or the product of one order two terms (xy) and the square of order 1 terms $(y+z)$.

Problem 9

Same as problem 6, the derivative and Hessian are

$$
\begin{aligned}
Df(x,y) &= (-4x^3+8x^2, -4y^3+12y^2) \\
Hf(x,y) &= \begin{bmatrix} -12x^2+16x & 0 \\ 0 & -12y^2+24y \end{bmatrix}.
\end{aligned}
$$

By setting $Df=0$, we have 4 critical points (0,0), (0,3), (2,0) and (2,3). For the first 3, $\det[Hf]=0$, so they can't be classified. For the last critical point, $\det[Hf]=576>0$, and $f_{xx}=-16<0$, so it's a local maximum.

For the global maximum, by plug into the 4 critical points, we know it attains its maximum at

$$f(2,3)=\frac{97}{3}.$$

Problem 10

Since $c=2s-a-b$, so we turn this problem into maximize

$$S(a,b)=A^2(a,b)=s(s-a)(s-b)(a+b-s)$$

with the constraint $s>0$ is some constant. So we only need to assume s is some constant, and find the global maximum for A^2. The derivative of S is

$$DS(a,b)=(s(s-b)(2s-2a-b), s(s-a)(2s-a-2b))$$

By setting it to 0, we have several solutions. If $b=s$, then we need either $a=0$ or $a=s$, which doesnt make sense because we don't want either side of the trianble to have length 0. So the only solution is $a = b = \frac{2s}{3}$. In this case c is also $\frac{2s}{3}$. This indicates that the maximum area is attained by equilateral triangle.

Problem 11

We can linearly expande the function A around (2,1)

$$\begin{aligned}
A(r,h) &= \pi r\sqrt{r^2+h^2} \\
&= \pi[(r-2)+2]\sqrt{[(r-2)+2]^2+[(h-1)+1]^2} \\
&\approx \pi[(r-2)+2]\sqrt{4(r-2)+2(h-1)+5} \\
&= \sqrt{5}\pi[(r-2)+2]\sqrt{1+\frac{4}{5}(r-2)+\frac{2}{5}(h-1)} \\
&\approx \sqrt{5}\pi[(r-2)+2][1+\frac{2}{5}(r-2)+\frac{1}{5}(h-1)] \\
&\approx \sqrt{5}\pi[2+\frac{9}{5}(r-2)+\frac{2}{5}(h-1)].
\end{aligned}$$

In the fifth line we are using the Taylor expansion of $\sqrt{1+x} \approx 1 + \frac{x}{2} + o(x)$. Now we plugin (1.98,1.03)

$$A(1.98, 1.03) \approx \sqrt{5}\pi[2 + 1.8 \cdot (-0.02) + 0.4 \cdot (0.03)] = 4.418\pi.$$

It's pretty close to the true value $A(1.98, 1.03) = 4.419\pi$.