114 SPRING 2011: Calculus

MIDTERM II (WITH SOLUTIONS)

PROBLEM 1: (10 pts) Which one of the following is the equation of the plane at $x = 7$, $y = 3$, and $z = -4$ that is tangent to a sphere of radius 5 centered at $x = 3$, $y = 0$, and $z = -4$?

(A) $8x + 6y = 0$

(B) $3x - 4z = 25$

(C) $14x + 6y - 8z = 138$

$\boxed{\text{(D)}}$ $4x + 3y = 37$

(E) $3x - 4z = 37$

(F) $8x + 6y = 25$

SOLUTION:

The sphere of radius 5 centered at $(3, 0, -4)$ is given by the equation

$$f(x, y, z) = (x - 3)^2 + y^2 + (z + 4)^2 = 25$$

A normal to it at the point $(7, 3, -4)$ is given by the gradient

$$\nabla f(7, 3, -4) = \langle 2(x - 3), 2y, 2(z + 4)\rangle\Big|_{(7,3,-4)} = \langle 8, 6, 0\rangle$$

Hence the equation of the normal plane we are looking for is

$$\langle x - 7, y - 3, z + 4\rangle \cdot \langle 8, 6, 0\rangle = 0 \quad \Rightarrow \quad 8x + 6y = 74$$

PROBLEM 2: (10 pts) Given that

$$f\begin{pmatrix} u \\ v \end{pmatrix} = u^2\ln(v) + \frac{u}{v} \quad ; \quad g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2xy \\ \cos(y) \end{pmatrix}$$

(1) Compute the derivative of $f(g(x,y))$ evaluated at $x = 1$ and $y = 0$: please use the chain rule and matrices. (2) If $x = 1$ and is changing at the rate of -1 and $y = 0$ and is changing at the rate of $+3$, what is the rate of change of the output of $f(g(x,y))$? Record this as your answer. Please use matrices, vectors, and the chain rule to derive your answers (though you may check your work with any other method, if you wish).

(A) 0

(B) -1

(C) $\ln 2$

(D) $3\ln 2$

(E) 3

$\boxed{\text{(F)}}$ 6

SOLUTION:

(1) We have

$$[Df(u,v)] = \begin{pmatrix} 2u\ln(v) + 1/v & u^2/v - u/v^2 \end{pmatrix}$$

$$[Dg(x,y)] = \begin{pmatrix} 2y & 2x \\ 0 & -\sin y \end{pmatrix}$$

Denoting $h = f \circ g$, the chain rule then gives

$$\begin{aligned} [Dh(1,0)] &= [Df(g(1,0))][Dg(1,0)] = [Df(0,1)][Dg(1,0)] \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \end{pmatrix} \end{aligned}$$

i.e., $h_x(1,0) = 0$ and $h_y(1,0) = 2$.

(2) The total rate of h is

$$\dot{h} = h_x(1,0)\,\dot{x} + h_y(1,0)\,\dot{y} = 0 \cdot (-1) + 2 \cdot 3 = 6$$

PROBLEM 3: (10 pts) The critical points of

$$f(x,y) = x^4 - 4xy + y^4$$

consist of the following:

(A) A saddle at $(0,0)$.

$\boxed{\text{(B)}}$ Minima at $(-1,-1)$ and $(1,1)$; and a saddle at $(0,0)$.

(C) Saddles at $(-1,-1)$ and $(1,1)$; and a minimum at $(0,0)$.

(D) Maxima at $(-1,-1)$ and $(1,1)$; and a saddle at $(0,0)$.

(E) Minima at $(-1,-1)$, $(-1,1)$, $(1,-1)$, and $(1,1)$; and a saddle at $(0,0)$.

(F) A minimum at $(0,0)$.

SOLUTION:

The locus of critical points of f is given by the vanishing of its gradient:

$$\nabla f = \langle 4x^3 - 4y, -4x + 4y^3 \rangle = 0 \quad \Rightarrow \quad \begin{cases} y = x^3 \\ x = y^3 \end{cases}$$

Eliminating y between these equations, we get $x = x^9$, with solutions $x = 0$, $x = 1$ and $x = -1$, giving three critical points: $(0,0)$, $(1,1)$ and $(-1,-1)$. To classify them, we look at the Hessian matrix of f:

$$H(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

Notice that $\det H(f) = 144x^2y^2 - 16$ and tr $H(f) = 12(x^2 + y^2)$. At both $(1,1)$ and $(-1,-1)$ the determinant and the trace are positive, so these points are minima of f. The origin is a saddle point, for there the determinant of the Hessian matrix is negative.

PROBLEM 4: (10 pts) Given the curve in the plane defined by the level set

$$F(x,y) = x\cos(y) - y\cos(x) - 2\pi = 0$$

Compute the tangent line to this curve at the point $(4\pi, 2\pi)$. Show all your work!

(A) $y = x + 2\pi$

(B) $y = -x + 4\pi$

(C) $x = 4\pi$

(D) $y = -x + 2\pi$

(E) $y = x + 4\pi$

$\boxed{\text{(F)}}$ $y = x - 2\pi$

SOLUTION:

The gradient of F at $(4\pi, 2\pi)$ gives a vector that is normal to the curve at that point:

$$\nabla F(4\pi, 2\pi) = \langle \cos y + y\sin x, -x\sin y - \cos x\rangle\Big|_{(4\pi,2\pi)} = \langle 1, -1\rangle$$

Any vector perpendicular to it is then tangent to the curve. For example, we can take the vector $\langle 1, 1\rangle$. We write the parametric equations of the tangent line we are looking for as

$$\vec{r}(t) = \langle 4\pi, 2\pi\rangle + t\langle 1, 1\rangle$$

Eliminating t between the equations for the components, we obtain $x - 4\pi = y - 2\pi$.

Another derivation of this equation is the following, reminiscent of the way we find the equation of a plane given a point and a normal vector: a point (x, y) is in our tangent line if the vector joining the points $(4\pi, 2\pi)$ and (x, y) is perpendicular to the normal vector $\nabla F(4\pi, 2\pi) = \langle 1, -1\rangle$, i.e.,

$$\langle x - 4\pi, y - 2\pi\rangle \cdot \langle 1, -1\rangle = 0$$

PROBLEM 5: (10 pts) Inscribe a rectangle with sides parallel to the x and y axes in the ellipse given by

$$f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Using the method of Lagrange multiplier(s), find the maximal area of such an inscribed rectangle.

(A) 0

(B) $\sqrt{2}ab$

$\boxed{\text{(C)}}$ $2ab$

(D) $\frac{1}{2}a^2b^2$

(E) $4ab$

(F) a^2b^2

SOLUTION:

The choice of a single point (x, y) in the part of the ellipse contained in the first quadrant of the plane specifies completely an inscribed rectangle with sides parallel to the axes: the other vertices must be the points $(-x, y)$, $(x, -y)$ and $(-x, -y)$. The area of such a rectangle is $A = 4xy$. To find the maximum of this area subject to the given constraint we need to solve the following system of equations:

$$\begin{cases} \nabla A + \lambda \nabla f = 0 \\ f = 1 \end{cases} \qquad \Rightarrow \qquad \begin{cases} 4y + 2\lambda x/a^2 = 0 \\ 4x + 2\lambda y/b^2 = 0 \\ x^2/a^2 + y^2/b^2 = 1 \end{cases}$$

Notice that we want the solution that has $x > 0$ and $y > 0$, since we chose (x, y) in the first quadrant.

We can isolate y in the first equation and substitute in the second one to get:

$$x\left(2 - \frac{\lambda^2}{2a^2b^2}\right) = 0$$

Since x cannot be zero (that would give a "rectangle" of zero area), it must be that $\lambda = \pm 2ab$. For the choice of the positive sign, the first equation in the original system of equations yields $y = -bx/a$. But any solution of the latter has x and y of different sign! Hence the solution we are looking for must have $\lambda = -2ab$, in which case $y = bx/a$. Substituting this into the equation of the constraint, we find

$$2\,\frac{x^2}{a^2} = 1$$

whose only positive solution is $x = a/\sqrt{2}$; the corresponding value of y is then $y = b/\sqrt{2}$, and the area function takes the value $A = 4\left(a/\sqrt{2}\right)\left(b/\sqrt{2}\right) = 2ab$.

PROBLEM 6: (10 pts) Using the method of Lagrange multipliers, set up but **do not solve** the equations to determine the point in $\mathbb{R}^3$ on the paraboloid $x = -y^2 - 2z^2$ **and** the point on the plane $2x + y + z = 7$ which are closest to each other. I will grade this problem based on whether you write down the correct equations to be solved. Your multi-choice answer is this: how many equations (and thus variables, including any Lagrange multipliers) do you have?

Hint 1: As with other problems, set it up to minimize the square of the distance, not the true distance. Hint 2: I do not want the minimal distance — I want the **points** on the plane and paraboloid at which the distance is minimized. Hint 3: Use (x, y, z) coordinates for the point on the plane and (u, v, w) coordinates for the point on the paraboloid. Hint 4: DO NOT SOLVE THESE EQUATIONS! Just set the problem up.

(A) 3

(B) 4

(C) 5

(D) 6

(E) 7

$\boxed{\text{(F)}}$ 8

SOLUTION:

The square of the distance between the points (x, y, z) and (u, v, w) is given by

$$f(x, y, z, u, v, w) = (x - u)^2 + (y - v)^2 + (z - w)^2$$

The constraints are

$$\begin{aligned} g(x, y, z, u, v, w) &= x + y^2 + 2z^2 = 0 \\ h(x, y, z, u, v, w) &= 2u + v + w = 7 \end{aligned}$$

The minimum of $f(x, y, z, u, v, w)$ subject to the above constraints will be reached at some pair of points $\{(x, y, z), (u, v, w)\}$ among the solutions of the following system of equations:

$$\begin{cases} \nabla f + \lambda \nabla g + \mu \nabla h = 0 \\ g = 0 \\ h = 7 \end{cases} \qquad \Rightarrow \qquad \begin{cases} 2(x - u) + \lambda = 0 \\ 2(y - v) + 2\lambda y = 0 \\ 2(z - w) + 4\lambda z = 0 \\ -2(x - u) + 2\mu = 0 \\ -2(y - v) + \mu = 0 \\ -2(z - w) + \mu = 0 \\ x + y^2 + 2z^2 = 0 \\ 2u + v + w = 7 \end{cases}$$

PROBLEM 7: (10 pts) The Taylor expansion of $f(x,y,z) = e^{x+z}\sin(xy)\cos(yz)$ about $x = y = z = 0$ up to and including terms of order 4 is...

(A) $xy + x^2yz$

(B) $1 + xy + \frac{1}{2}x^2yz$

(C) $xy + x^2y + xyz$

(D) $xy + x^2z + xyz$

(E) $2 + x + z + \frac{1}{2}x^2 + xz + \frac{1}{2}z^2 + xy + \frac{1}{2}y^2z^2$

$\boxed{\text{(F)}}$ $xy + x^2y + xyz + \frac{1}{2}x^3y + x^2yz + \frac{1}{2}xyz^2$

SOLUTION:

Since $x + z = 0$ at the origin, we can use our knowledge of the Taylor expansion of the exponential function around 0 to get

$$e^{x+z} = 1 + (x+z) + \frac{(x+z)^2}{2} + \frac{(x+z)^3}{6} + \frac{(x+z)^4}{24} + \cdots$$

The higher order terms are all of order 5 or more, so we do not need to consider them. Similarly, we obtain

$$\cos(yz) = 1 - \frac{(yz)^2}{2} - \cdots, \qquad \sin(xy) = xy - \cdots$$

All that is left to do is multiply these expressions, keeping only the terms up to order 4. Since the only term in the series for sine that has order less or equal than 4 is xy, which has order 2, it is enough to consider products of terms from the exponential and cosine series that have order less or equal than 2:

- taking 1 in both the exponential and cosine series, we get xy;
- taking $x + z$ in the first series and 1 in the second, we get $x^2y + xyz$;
- $(x+z)^2/2$ from the exponential and 1 from the cosine yields $\frac{1}{2}x^3y + x^2yz + \frac{1}{2}xyz^2$;
- all other possible products lead to terms of order strictly greater than 4.

PROBLEM 8: (10 pts) If I tell you that the Taylor series of $f(x,y,z)$ about $(0,0,0)$ equals

$$f(x,y,z) = 4 - x^2 + xy - y^2 - \frac{1}{2}z^2 + \frac{1}{3}x^3 + x^2 z + \frac{1}{2}yz^2 + \text{H.O.T.}$$

what can you tell me about the behavior of the function near the origin? Please do the following: (0) Explain why this is a critical point (1-line answer!); (1) isolate the second order terms from the Taylor series; (2) factor that quadratic form into the sum/difference of squares; (3) draw your conclusions. You will be graded based on the **work you show**, not merely on the answer you choose: show/explain all steps!

(A) The origin is a local minimum.

$\boxed{\text{(B)}}$ The origin is a local maximum.

(C) The origin is a saddle.

(D) The origin is a degenerate critical point.

SOLUTION:

(0) The origin is a critical point because the partial derivatives $f_x(0,0,0)$ and $f_y(0,0,0)$ vanish; we know this because these derivatives are the coefficients of the linear part of the Taylor expansion around the origin.

(1) By inspection: $-x^2 + xy - y^2 - \frac{1}{2}\,z^2$.

(2) Completing the square, we have

$$-x^2 + xy = -\left(x - \frac{1}{2}\,y\right)^2 + \frac{1}{4}\,y^2$$

so we can rewrite the second order part of the Taylor expansion as

$$-x^2 + xy - y^2 - \frac{1}{2}\,z^2 = -\left(x - \frac{1}{2}\,y\right)^2 - \frac{3}{4}\,y^2 - \frac{1}{2}\,z^2$$

(3) The coefficients of the terms in the quadratic form above are all negative, so the origin is maximum of f.

PROBLEM 9: (10 pts) *Chemotaxis* is the mechanism whereby certain cells (bacteria, e.g.) move according to change in some ambient stimulus (food supply, light, glucose, or other chemical concentration). If one models the amount of stimulus as a function $S(x, y)$ depending on position (x, y), then the cell may be said to move **always** in the direction of **maximal increase** of S.

In a specific example, say that $S(x, y) = 10 - x^2 + 3xy - 3y^2$. Tell me (a) the direction a bacterium at $x = 2, y = 1$ moves; and (b) where does the bacterium eventually go? Give clear work/reasons for your answer!

(A) (a) it moves in the direction $\hat{\imath}$; (b) it goes to the origin

$\boxed{\text{(B)}}$ (a) it moves in the direction $-\hat{\imath}$; (b) it goes to the origin

(C) (a) it moves in the direction $\hat{\imath}$; (b) it goes very far from the origin

(D) (a) it moves in the direction $-\hat{\imath}$; (b) it goes very far from the origin

(E) (a) it moves in the direction $\hat{\imath}$; (b) not enough information

(F) (a) it moves in the direction $-\hat{\imath}$; (b) not enough information

SOLUTION:

(a) At any given point, the direction of maximum increase of the function S is given by the direction of the gradient vector ∇S. At the point $(2, 1)$, we have

$$\nabla S(2, 1) = \langle -2x + 3y, 3x - 6y\rangle\Big|_{(2,1)} = \langle -1, 0\rangle$$

(b) If a bacterium keeps moving in the direction of maximum increase it should end up at a maximum of the function S, if such a point exists. The origin is the only critical point, for

$$\nabla S = \langle -2x + 3y, 3x - 6y\rangle = 0$$

has onyl $x = 0$ and $y = 0$ as solution. The Hessian matrix of S there,

$$H(S)(0, 0) = \begin{pmatrix} -2 & 3 \\ 3 & -6 \end{pmatrix}$$

has positive determinant and negative trace, confirming that the origin is indeed a maximum of S.

PROBLEM 10: (10 pts) A glass bottle of wine is (approximately) a cylinder with interior dimensions of height 27cm and radius 3cm. The manufacturer decides to save money by keeping the outer dimensions the same and increasing the thickness of the glass, thereby decreasing both interior height and radius by a fixed amount ϵ. Use a linear approximation to estimate the change in volume as a function of ϵ.

(A) volume decreases by $81\pi\epsilon$

(B) volume decreases by $151\pi\epsilon$

$\boxed{\text{(C)}}$ volume decreases by $171\pi\epsilon$

(D) volume decreases by $243\pi\epsilon$

(E) volume decreases by $407\pi\epsilon$

(F) volume decreases by $414\pi\epsilon$

SOLUTION:

Let h be the interior height of the bottle and r its inner radius. The volume of the bottle is given by $V = \pi r^2 h$. If we change the interior height by $\dot{h}$ and the interior radius by $\dot{r}$, the change in volume is given to first order by:

$$\dot{V} = V_r \dot{r} + V_h \dot{h} = 2\pi r h \dot{r} + \pi r^2 \dot{h}$$

Increasing the thickness of the glass by ϵ means $\dot{h} = \dot{r} = -\epsilon$, so

$$\dot{V} = 2\pi \cdot 3 \cdot 27 \cdot (-\epsilon) + \pi \cdot 3^2 \cdot (-\epsilon) = -171\pi\epsilon$$