MATH 114 Sample Midterm 3 Solution

April 11, 2011

Problem 1

Solution:
First we can isolate y'

$$y' - \frac{2}{x}y = x.$$

We have $p(x) = -\frac{2}{x}$, $Q(x) = x$. The integrating factor is

$$I(x) = e^{\int P(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = x^{-2}.$$

Using the formula for general solution

$$y(x) = \frac{\int Q(x)I(x)dx + C}{I(x)} = \frac{\int x^{-1}dx + C}{x^{-2}} = x^2(\log x + C).$$

With initial condition $y(1) = e$, the special solution is $y(x) = x^2(\log x + e)$. So $y(e)$ is

$$y(e) = e^2(\log e + e) = e^2 + e^3.$$

Problem 2

Solution:
Same as in problem 1. We first isolate y'

$$y' - xy = \frac{\cos x}{x} \tag{1}$$

Then $P(x) = -x$, $Q(x) = \frac{\cos x}{x}$. The integrating factor is

$$I(x) = e^{\int P(x)dx} = e^{\int -xdx} = e^{-\frac{x^2}{2}}.$$

By multiplying $I(x)$ to the left side of equation (1)

$$e^{-\frac{x^2}{2}}(y' - xy) = e^{-\frac{x^2}{2}}\frac{dy}{dx} + y\frac{d}{dx}(e^{-\frac{x^2}{2}}) = \frac{d}{dx}(e^{-\frac{x^2}{2}}y).$$

Where the last equation follows from the product rule.

Problem 3

Solution:

(Please figure out the region by yourself) The original version of the region is $R = \{(x,y)|0 \leq x \leq 3, \sqrt{x/3} \leq y \leq 1\}$. The other version is $R = \{(x,y)|0 \leq y \leq 1, 0 \leq x \leq 3y^2\}$. So the other way to interpret the original double integral is

$$\int_0^1 \int_0^{3y^2} e^{y^3}\, dxdy = \int_0^1 3y^2 e^{y^3}\, dy = \int_0^1 e^{y^3} d(y^3) = e^{y^3}\Big|_0^1 = e - 1.$$

Problem 4

Solution:

The first shpere under Cartesian coordinates is ($\rho^2 = \rho\cos(\phi)$) $x^2+y^2+z^2 = z$. By completing square, we have $x^2 + y^2 + (z - \frac{1}{2})^2 = (\frac{1}{2})^2$. So the region looks like a small ball being removed from a large hemisphere. Since $z \geq 0$, then the region in shperical coordinates is $E = \{(\rho, \phi, \theta)|0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2, \cos\phi \leq \rho \leq 2\}$. The triple integral will give us the volume of the region E

$$\begin{aligned}
V(E) &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_{\cos\phi}^{2} 1 \cdot \rho^2 \sin(\phi) \cdot d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{\rho^3}{3}\Big|_{\rho=\cos(\phi)}^{2} \sin(\phi) d\phi d\theta \\
&= \frac{8}{3}\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin(\phi) d\phi d\theta - \frac{1}{3}\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos^3(\phi)\sin(\phi) d\phi d\theta
\end{aligned}$$

Where the first term is actually the volume of the hemisphere.

$$\frac{8}{3}\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin(\phi) d\phi d\theta = \frac{8}{3}\int_0^{2\pi} [-\cos(\phi)]|_{\phi=0}^{\frac{\pi}{2}}\, d\theta = \frac{16\pi}{3}.$$

While the second term is the volume of the removed small ball. We have $\cos^3(\phi)\sin(\phi)d\phi = -\cos^3(\phi)d(\cos(\phi)) = -\frac{1}{4}d(\cos^4(\phi))$, so we can use the change of variable to evalue the inner integral

$$\frac{1}{3}\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos^3(\phi)\sin(\phi) d\phi d\theta = \frac{1}{3}\int_0^{2\pi} [-\frac{1}{4}\cos^4(\phi)]\Big|_{\phi=0}^{\frac{\pi}{2}} d\theta = \frac{\pi}{6}.$$

So the original integral is

$$V(E) = \frac{16\pi}{3} - \frac{\pi}{6} = \frac{31\pi}{6}.$$

Problem 5

Solution: (Updated)

(a) Assume the radius is 1 without loss of generality. The solid lies above the 45° north latitude line is enclosed by the surfaces $z = \sqrt{1 - x^2 - y^2}$ (top of hemisphere) and $z=\sqrt{2}/2$. By letting the first surface higher than the second

$$\begin{aligned}\sqrt{1-x^2-y^2} &\geq \frac{\sqrt{2}}{2}\\ x^2+y^2 &\leq \frac{1}{2},\end{aligned}$$

we have the region over which the double integral should be set up. To evaluate the volume

$$\iint\limits_{x^2+y^2\leq 1/2} \sqrt{1-x^2-y^2}-\frac{\sqrt{2}}{2}\,dxdy.$$

The first part can be evaluated as

$$\begin{aligned}&\iint\limits_{x^2+y^2\leq 1/2} \sqrt{1-x^2-y^2}\,dxdy\\ =& \int_0^{2\pi}\int_0^{\frac{\sqrt{2}}{2}} \sqrt{1-r^2}r\,drd\theta\\ =& 2\pi\int_0^{\frac{\sqrt{2}}{2}} \sqrt{1-r^2}\frac{1}{2}d(r^2)\\ =& 2\pi\left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}}\right]\Bigg|_0^{\frac{\sqrt{2}}{2}}\\ =& \frac{2\pi}{3}-\frac{\sqrt{2}\pi}{6}.\end{aligned}$$

The second part describes a cylinder with base area $\pi/2$ and height $\sqrt{2}/2$, so the volume is $\sqrt{2}\pi/4$. Then the total volume is $\frac{2\pi}{3}-\frac{5\sqrt{2}\pi}{12}$.

Hence the proportion is

$$p=\frac{\frac{2\pi}{3}-\frac{5\sqrt{2}\pi}{12}}{\frac{4\pi}{3}}=\frac{8-5\sqrt{2}}{16}.$$

(b) Under the (same!) appropriate coordinate transformation, the volume of northern region and the whole earth are simutaneouly multiplied by the same constant $\det(J)=\lambda^2$. For both of them, the volume is increasing at the same rate, and the proportion stays the same.

Problem 6

Solution:
The region is just the first quadrant, with polar coordinates $R=\{(r,\theta)|0\leq$

$r < \infty, 0 \le \theta \le \pi/2\}$. The double integral can be rewritten as

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2} dxdy \\
= & \int_0^{\frac{\pi}{2}} \int_0^\infty \frac{1}{(1+r^2)^2} rdrd\theta \\
= & \frac{\pi}{2} \int_0^\infty \frac{1}{(1+r^2)^2} \frac{1}{2} d(1+r^2) \\
= & \frac{\pi}{4} \left[-\frac{1}{1+r^2} \right] \Big|_{r=0}^{\infty} \\
= & \frac{\pi}{4},
\end{aligned}$$

where the third equation is because $d(1+r^2) = 2rdr$.

Problem 7

Solution:

By eliminating r in two equations, we know $4\cos\theta = \sec\theta$, the solution is $\theta = \pm\arccos(\frac{1}{2}) = \pm\frac{\pi}{3}$. So the region can be interpreted as $R = \{(r,\theta) | \frac{\pi}{3} \le \theta \le \frac{\pi}{3}, \sec\theta \le r \le 4\cos\theta\}$. The double integral to evaluate the area is (with a little help from integration table)

$$\begin{aligned}
\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{\sec\theta}^{4\cos\theta} rdrd\theta &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 8\cos^2\theta - \frac{1}{2\cos^2\theta} d\theta \\
&= \left[4\theta + 2\sin(2\theta) - \frac{1}{2}\tan\theta \right] \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\
&= \frac{8\pi}{3} + 2\sqrt{3} - \sqrt{3} \\
&= \frac{8\pi}{3} + \sqrt{3}
\end{aligned}$$

To give you some geometric sense, the first equation is a circle with center $(2,0)$ and radius 2. The second equation is the straight line $x = 1$.

Problem 8

Solution:

First of all, the volume of unit cube is always 1 (That's why it's called *unit*). Then the average of square distance to the origin of a unit cube in $\mathbf{R}^n$ is just the n-ple integral

$$\begin{aligned}
\int_{-0.5}^{0.5} \cdots \int_{-0.5}^{0.5} \sum_{i=1}^{n} x_i^2 dx_1 \cdots dx_n &= \int_{-0.5}^{0.5} \cdots \int_{-0.5}^{0.5} n x_1^2 dx_1 \cdots dx_n \\
&= \int_{-0.5}^{0.5} n x_1^2 dx_1 = n/12.
\end{aligned}$$

The first equation is due to symmetry over indices. The second equation is due to Fubini's theorem and evaluating all the irrelevant integrals $\int_{-0.5}^{0.5} 1 dx_j = 1$ with $j > 1$.

Problem 9

Solution:

By eliminating z, we have the area on xy-space. $5 - x^2 - y^2 \geq 4x^2 + 4y^2$ since the first paraboloid is on top of the second one. By solving this inequality we have $x^2 + y^2 \leq 1$. So the volume is

$$\begin{aligned}
& \iint\limits_{x^2+y^2\leq 1} 5 - x^2 - y^2 - (4x^2 + 4y^2)\, dxdy \\
= & \int_0^{2\pi} \int_0^1 (5 - 5r^2) r\, dr d\theta \\
= & 2\pi \left[\frac{5}{2} r^2 - \frac{5}{4} r^4 \right] \Big|_0^1 \\
= & \frac{5\pi}{2}
\end{aligned}$$

Problem 10

Solution: (Updated)

The original region of double integral is a triangle with vertices $(0, 0)$, $(2, 0)$ and $(2/3, 2/3)$ in xy-coordinates. Now if we use the linear transformation $u = x + 2y$ and $v = x - y$. The Jacobian and its determinant is

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}, \det(J) = -3.$$

The new uv-coordinates for those vertices are $(0, 0)$, $(2, 2)$ and $(2, 0)$. The linear transformation will only shear or translate the region. So the new region must be the triangle with those 3 new vertices. So we can choose our new region as $R = \{(u, v) | 0 \leq u \leq 2, 0 \leq v \leq u\}$. The new double integral is

$$\begin{aligned}
& \int_0^2 \int_0^u ue^{-v} \left| \det(\frac{\partial(x, y)}{\partial(u, v)}) \right| dvdu \\
= & \int_0^2 \int_0^u ue^{-v} \left| \det(J^{-1}) \right| dvdu \\
= & \frac{1}{3} \int_0^2 u(1 - e^{-u}) du \\
= & \frac{1}{3} \left[\frac{u^2}{2} + ue^{-u} + e^{-u} \right] \Big|_0^2 \\
= & \frac{1}{3} + e^{-2}.
\end{aligned}$$

Please note that you would need to do integration by parts (or look at an integration table) in the middle of this.

Problem 11

Solution:

The region can be descried as $E = \{(x,y,z)|x \in [0,2], y \in [0,\sqrt{x}], z \in [0, 4-x^2]\}$. So the mass of the solid is

$$\begin{aligned}
m &= \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} \rho(x,y,z)\, dzdydx = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} xy\, dzdydx \\
&= \int_0^2 \int_0^{\sqrt{x}} [z]|_0^{4-x^2}\, xy\, dydx = \int_0^2 \int_0^{\sqrt{x}} (4x - x^3) y\, dydx \\
&= \int_0^2 (4x - x^3) \left[\frac{y^2}{2}\right]\Big|_0^{\sqrt{x}} dx = \int_0^2 (2x^2 - \frac{x^4}{2})\, dx \\
&= \left[\frac{2x^3}{3} - \frac{x^5}{10}\right]\Big|_0^2 = \frac{32}{15}.
\end{aligned}$$

And the total x coordinate is

$$\begin{aligned}
m_x &= \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} x\rho(x,y,z)\, dzdydx = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} x^2 y\, dzdydx \\
&= \int_0^2 \int_0^{\sqrt{x}} [z]|_0^{4-x^2}\, x^2 y\, dydx = \int_0^2 \int_0^{\sqrt{x}} (4x^2 - x^4) y\, dydx \\
&= \int_0^2 (4x^2 - x^4) \left[\frac{y^2}{2}\right]\Big|_0^{\sqrt{x}} dx = \int_0^2 (2x^3 - \frac{x^5}{2})\, dx \\
&= \left[\frac{x^4}{2} - \frac{x^6}{12}\right]\Big|_0^2 = \frac{8}{3}.
\end{aligned}$$

The total y coordinate is

$$\begin{aligned}
m_y &= \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} y\rho(x,y,z)\, dzdydx = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} xy^2\, dzdydx \\
&= \int_0^2 \int_0^{\sqrt{x}} [z]|_0^{4-x^2}\, xy^2\, dydx = \int_0^2 \int_0^{\sqrt{x}} (4x - x^3) y^2\, dydx \\
&= \int_0^2 (4x - x^3) \left[\frac{y^3}{3}\right]\Big|_0^{\sqrt{x}} dx = \int_0^2 (\frac{4x^{2.5}}{3} - \frac{x^{4.5}}{3})\, dx \\
&= \left[\frac{4x^{3.5}}{10.5} - \frac{x^{5.5}}{16.5}\right]\Big|_0^2 = \frac{64\sqrt{2}}{231}.
\end{aligned}$$

The total z coordinate is

$$\begin{aligned}
m &= \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} z\rho(x,y,z)\, dzdydx = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} xyz\, dzdydx \\
&= \int_0^2 \int_0^{\sqrt{x}} \left[\frac{z^2}{2}\right]\Big|_0^{4-x^2} xy\, dydx = \int_0^2 \int_0^{\sqrt{x}} (8x - 4x^3 + \frac{x^5}{2})y\, dydx \\
&= \int_0^2 (8x - 4x^3 + \frac{x^5}{2}) \left[\frac{y^2}{2}\right]\Big|_0^{\sqrt{x}} dx = \int_0^2 (4x^2 - 2x^4 + \frac{x^6}{4})\, dx \\
&= \left[\frac{4x^3}{3} - \frac{2x^5}{5} + \frac{x^7}{28}\right]\Big|_0^2 = \frac{256}{105}.
\end{aligned}$$

Hence the center of mass is at $\frac{1}{m}(m_x, m_y, m_z) = (\frac{5}{4}, \frac{10\sqrt{2}}{77}, \frac{8}{7})$.

Problem 12

Solution:

The original region can be regarded as a triangle with vertices $(0,0)$, $(R,0)$ and (R,R) under xy-coordinates with $R \to \infty$. With the same idea as problem 10, if we apply the transformation $u = x - y, v = y$. The Jacobian and its determinant is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \det(J) = 1.$$

The vertices are transformed to $(0,0)$, $(R,0)$ and $(0,R)$ under uv-coordinates. However, as $R \to \infty$, the limit of the region is the first quadrant. Hence the new double integral is

$$\int_0^\infty \int_0^\infty e^{-su-sv} f(u,v) dudv$$