Some summation identities and their computer proofs

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1 The recurrences

Let $m, q \geq 0$ be given integers. We define

$$p_r(y) = \binom{q}{r}(y+1)^{2r}(y-1)^{2q-2r},$$

and $P_{m,q}(y) = \sum_{r=0}^{q} a_r y^r$, where

$$a_r = \binom{q}{r}\prod_{j=1}^{r}\frac{2m+2j-1}{2m+2q-2j+1} = (-1)^r\binom{q}{r}\frac{(m+\frac{1}{2})_r}{(\frac{1}{2}-m-q)_r},$$

and $(a)_r = a(a+1)\cdots(a+r-1)$ is the usual rising factorial.

In this note, we prove the following identity which arose from a trigonometric integral in the study [1] of the Radon transform on Grassmannians:

$$\sum_{r=0}^{q}\binom{2m+q}{m+r}p_r(y) = \binom{2m+2q}{m+q}P_{m,q}(y^2). \tag{1}$$

Indeed, if u is the complex-valued function in two variables defined by

$$u(x,t) = t\sin x - i\cos x,$$

for $x, t \in \mathbb{R}$, the function $F_{m,q}$ defined by

$$F_{m,q}(t) = \frac{1}{2\pi}\int_0^{2\pi}(u^{2m+q}\,\bar{u}^q)(x,t)\,dx, \tag{2}$$

for $t \in \mathbb{R}$, is real-valued and satisfies

$$F_{m,q}(t) = \frac{1}{4^{m+q}}\,(t^2-1)^m\sum_{r=0}^{q}\binom{2m+q}{m+r}p_r(t), \tag{3}$$

for $t \in \mathbb{R}$. Indeed, if we view the integrand of (2) as a polynomial in e^{ix} and e^{-ix}, the constant term of this polynomial is equal to the right side of (3). By evaluating the integral (3) in special cases, the first two authors were led to the identity (1).

We now begin our verification of the identity (1). A first thought might be to equate the coefficients of like powers of y on both sides of (1) and prove the resulting identity. That would lead to proving that a certain double sum is equal to a certain explicit term. A better idea is to replace y by $(y+1)$ throughout. Then the factor $(y-1)^{2q-2r}$ in the definition of p_r becomes just a power of y and saves one summation. The result will be that we will want to prove the equality of two *single* summations, which is easy by Zeilberger's algorithm.

So replace y by $y+1$ in (1) and equate the coefficients of y^n on both sides. The result is that we need to prove the following identity:

$$\sum_r (-1)^r \binom{2m+2q}{m+q}\binom{2r}{n}\binom{q}{r}\frac{(m+\frac{1}{2})_r}{(\frac{1}{2}-m-q)_r} = 2^{2q-n}\sum_r \binom{2m+q}{m+r}\binom{q}{r}\binom{2r}{n+2r-2q}. \tag{4}$$

Zeilberger's algorithm, as implemented in the EKHAD package, handles this easily. Upon inputting the two summands above it finds that they both satisfy the same recurrence of order 2, namely they are both annihilated by the operator

$$\Omega(n,N) = (n-2q)(n+2m+1)+(3n^2+4nm-4nq-4mq+7n+4m-6q+4)N+2(n+2)(n+2m+2)N^2, \tag{5}$$

where N is the forward shift $Nf_n = f_{n+1}$. That fact, together with the fact, proven below, that the sums agree with each other when $n=0$ and $n=1$ finishes the proof.

The routine EKHAD also supplies short proofs of the assertions that the above sums satisfy the recurrence stated, by a standard procedure, namely the following.

To verify that a given sum $f(n) = \sum_r F(n,r)$ satisfies a given recurrence relation one needs two things-

1. the recurrence operator $\Omega(N,n)$, say, such that $\Omega(N,n)f(n)$ is allegedly 0.
2. a rational function $R(n,r)$, called the certificate.

Given those two things, one does the following proof, separately for each sum that you want to handle.

1. Define $G(n,r) = R(n,r)F(n,r)$.
2. Check that $\Omega(N,n)F(n,r) = G(n,r+1) - G(n,r)$.

   (This is a routine computation because there is no summation involved. Just take the *summand* $F(n,r)$ and apply the recurrence operator to it, and check that the result is as shown. To check that just cancel out all of the factorials on both sides and you'll be left with a polynomial identity that will trivially say that $0=0$.)

3. By summing the above telescoping relation, which you just verified, over r, and noting that Ω is independent of r, the right side telescopes to 0, and you have shown that the sum itself satisfies the given recurrence. $\square$

That's the whole proof procedure, and the only things that vary from one identity to another are the recurrence operator and the rational function R.

Now, for the sum on the left side of (4) the certificate $R(n,r)$ is

$$\frac{(2r-n-1)(2r-n)(2r-1-2m-2q)}{n+1},$$

and for the sum on the right side of that equation the certificate $R(n,r)$ is $(2q-n)(m+r)$; this concludes the proof that both sides of (4) satisfy the same recurrence relation of order 2.

2 The initial conditions

Since both sides satisfy this recurrence, to complete the proof we must show that the two sides are equal for, say, $n=0$ and $n=1$. That means that we must now prove the two identities,

$$\sum_r (-1)^r \binom{2m+2q}{m+q}\binom{q}{r}\frac{(m+\frac{1}{2})_r}{(\frac{1}{2}-m-q)_r} = 2^{2q}\sum_r \binom{2m+q}{m+r}\binom{q}{r}\binom{2r}{2r-2q}, \tag{6}$$

and

$$\sum_r (-1)^r \binom{2m+2q}{m+q} 2r \binom{q}{r}\frac{(m+\frac{1}{2})_r}{(\frac{1}{2}-m-q)_r} = 2^{2q-1}\sum_r \binom{2m+q}{m+r}\binom{q}{r}\binom{2r}{1+2r-2q}, \tag{7}$$

which are easier, in that one less parameter is involved, but are still non-trivial.

The left side of (6) actually evaluates in simple closed form (in fact, it is a case of Gauss's original ${}_2F_1$ identity). The clue to this is that Zeilberger's algorithm returns a recurrence of order 1, in the index m, that is satisfied by the left side, namely the operator

$$(2m+q+2)(2m+q+1)-(m+1)(m+q+1)M,$$

where M is a forward shift in m, annihilates the sum. This gives us the evaluation of the left side, viz.,

$$\sum_r (-1)^r \binom{2m+2q}{m+q}\binom{q}{r}\frac{(m+\frac{1}{2})_r}{(\frac{1}{2}-m-q)_r} = \binom{2m+2q}{m+q}\frac{(2m+1)_q}{(m+1/2)_q} = 4^q\binom{2m+q}{m}. \tag{8}$$

Likewise, for the left side of (7), the algorithm returns a first order recurrence, namely

$$(2m+q+2)(2m+q+1)-(m+1)(m+q+1)M,$$

annihilates that sum, which gives the closed form evaluation

$$\sum_r (-1)^r \binom{2m+2q}{m+q} 2r \binom{q}{r} \frac{(m+\frac{1}{2})_r}{(\frac{1}{2}-m-q)_r} = q\,4^q \binom{2m+q}{m}.$$

It remains to show that the right sides of (6) and (7) have the same evaluations. However, in each of those two right sides, there is only one value of the summation index that contributes a non-vanishing term, namely $r = q$, and the two sums are easily seen to be equal to the left hand sides shown above, completing the proof.

Reference

[1] J. Gasqui and H. Goldschmidt, Isospectral deformations of the Grassmannian of 3-planes in $\mathbb{R}^6$ (to appear).