

Take a positive integer n and repeat the following operations: if n is odd then replace n by $3n + 1$, whereas if n is even then replace n by $n/2$. For example if we start with $n = 34$ we obtain successively,

$$34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

The problem is to prove or disprove the assertion that *whatever n you start with, you’ll end up at $n = 1$ after a finite number of steps*. This problem has been unsolved since it was proposed in the 1930’s.

If $n \geq 2$ is an integer, then $S(n)$, the *successor* of n , is $n/2$ if n is even and is $3n + 1$ if n is odd. Say that m is a *good* integer if the chain

$$m \rightarrow S(m) \rightarrow S(S(m)) \rightarrow \dots,$$

that begins with m and proceeds by the successor operator, reaches 1 after finitely many steps. Such a chain, that begins with m and reaches 1 in finitely many steps, will be called an m -chain.

Define ϵ_m to be 1 if $m = 1$ or if $m \geq 2$ and m is good, and to be 0 otherwise. The $3x + 1$ problem is to prove or disprove the assertion that $\epsilon_m = 1$ for every positive integer m .

We have the following three relations among the ϵ ’s:

- (a) $\epsilon_1 = 1$, and
- (b) $\epsilon_{2k} = \epsilon_k$ for all $k \geq 1$, and
- (c) $\epsilon_{2k+1} = \epsilon_{6k+4}$ for all $k \geq 1$.

Indeed, if $\epsilon_{2k} = 1$, there is a $2k$ -chain $2k \rightarrow k \rightarrow \dots$, so $\epsilon_k = 1$ also. Conversely if $\epsilon_k = 1$ there is a k chain, and therefore a $2k$ -chain also since $S(2k) = k$, which proves (b). Similarly we prove (c).

Next define the generating function $f(t) = \sum_{k \geq 1} \epsilon_k t^k$. To discover the functional equation that f satisfies, we do the following: multiply (a) above by t , add to the sum over $k \geq 1$ of the result of multiplying (b) by t^{2k} , and add the sum over $k \geq 1$ of the result of multiplying (c) by t^{2k+1} . The result is that

$$f(t) = t + \sum_{k \geq 1} \epsilon_k t^{2k} + \sum_{k \geq 1} \epsilon_{6k+4} t^{2k+1} = f(t^2) + \sum_{k \geq 0} \epsilon_{6k+4} t^{2k+1}. \quad (1)$$

This is the first form of the functional equation. From this form it is easy to check that $f(t) = t/(1-t)$ satisfies the equation. It follows that *the $3x + 1$ problem has an affirmative*

answer if and only if equation (1) has a unique solution in the class of power series of the form $f(t) = t + O(t^2)$.

For a second form of the functional equation (1) we observe that for any power series $g(t) = \sum_{n \geq 0} b_n t^n$, we can pick out the subseries in which the exponents are congruent to u modulo v by means of the easily verified identity

$$\sum_{r \geq 0} b_{rv+u} t^{rv+u} = \frac{1}{v} \sum_{\omega^v=1} \omega^{-u} g(\omega t),$$

in which the sum extends over the v -th roots of unity.

Now if we replace t by t^3 in (1) we obtain

$$f(t^3) = f(t^6) + \frac{1}{t} \sum_{k \geq 0} \epsilon_{6k+4} t^{6k+4} = f(t^6) + \frac{1}{6t} \sum_{\omega^6=1} \omega^2 f(\omega t), \quad (2)$$

in which the sum is over the sixth roots of unity. Thus-

The $3x + 1$ problem has an affirmative answer iff the functional equation

$$f(t^3) = f(t^6) + \frac{1}{6t} \sum_{\omega^6=1} \omega^2 f(\omega t) \quad (3)$$

has a unique solution in the class of functions $f(t)$ s.t. $f(0) = 0$, $f'(0) = 1$, and $f(t) = \sum_{j \geq 1} \epsilon_j t^j$, ($\forall j : \epsilon_j = 0, 1$).

A proposition that is somewhat stronger than the $3x + 1$ problem is therefore the following.

Let f satisfy (3) and be holomorphic in the unit disk, with $f(0) = f'(0) = 0$. Then f vanishes identically.

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Suppose we represent $f(z)$ by the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

If we substitute this into (3) we get

$$\frac{1}{2\pi i} \int_C f(\zeta) d\zeta \left\{ \frac{1}{\zeta - z^3} - \frac{1}{\zeta - z^6} - \frac{1}{6z} \sum_{\omega^6=1} \frac{\omega^2}{\zeta - \omega z} \right\} = 0.$$

But it's easy to check that the inner sum is

$$\sum_{\omega^6=1} \frac{\omega^2}{\zeta - \omega z} = \frac{6z^4 \zeta}{\zeta^6 - z^6}.$$

To sum up, let's define the kernel function

$$K(\zeta, z) = \frac{z^3 - z^6}{(z^3 - \zeta)(z^6 - \zeta)} - \frac{z^3 \zeta}{\zeta^6 - z^6},$$

and the integral operator

$$(\mathcal{K}f)(z) = \frac{1}{2\pi i} \int_C K(\zeta, z) f(\zeta) d\zeta,$$

from $H \rightarrow H$, where H is the set of holomorphic functions on the unit disk which vanish at the origin.

The $3x+1$ problem is equivalent to the assertion that *the operator \mathcal{K} has a one-dimensional kernel*, namely the constant multiples of $z/(1-z)$.