Basis Partitions

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Abstract

We study *basis partitions*, introduced by Hansraj Gupta in 1978. For this family of partitions, we give a recurrence, a generating function, identities relating basis partitions to more familiar families of partitions, and a new characterization of basis partitions.

1 Introduction

In a 1978 paper [1], Hansraj Gupta introduced an interesting class of integer partitions called *basis partitions*. An integer partition is a basis partition if, in the class of all partitions with its rank vector (see below), its weight is minimum. A *partition* π of a positive integer n is a sequence of positive integers $(\pi_1, \pi_2, \ldots, \pi_l)$ satisfying $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_l$ and $\pi_1 + \pi_2 + \ldots + \pi_l = n$. We will call n the *weight* of π, and will write $n = |\pi|$. We write $P(n)$ for the set of all partitions of n, where $P(0)$ contains only the empty partition, λ.

For a partition $\pi = (\pi_1, \ldots, \pi_l)$, the associated *Ferrers graph* is the array of l rows of dots, where row i has π_i dots and rows are left justified. Let π' denote the conjugate partition $\pi' = (\pi'_1, \ldots, \pi'_m)$ where $m = \pi_1$ and π'_i is the number of dots in the i-th column of the Ferrers graph of π. The *Durfee square* of π is the largest square subarray of dots in the Ferrers graph of π. Let $d(\pi)$ denote the size (number of rows) of the Durfee square of π. As in [1], define the *rank vector* of π,

$$\mathbf{r}(\pi) = [\pi_1 - \pi'_1, \;\; \pi_2 - \pi'_2, \;\; \ldots, \;\; \pi_{d(\pi)} - \pi'_{d(\pi)}]$$

*Supported by the National Science Foundation through the CRA Distributed Mentor Project, 1995
†Supported in part by National Science Foundation Grant No. DMS 930 2505 and DIMACS
‡Supported in part by the Office of Naval Research

(10)	(4,4,1,1)
(8,2)	(3,3,3,1)
(7,3)	(3,3,1,1,1,1)
(6,2,2)	(2,2,2,2,2)
(6,4)	(2,2,2,2,1,1)
(5,5)	(2,2,2,1,1,1,1)
(4,3,3)	(2,2,1,1,1,1,1,1)
(4,2,2,2)	(1,1,1,1,1,1,1,1,1,1)

Figure 1: The basic partitions of 10.

to be the vector of length $d(\pi)$ whose entries are what Atkin [2] calls the *successive ranks* of π.

Note that a given rank vector is associated with infinitely many partitions. For example, the partitions $\alpha = (13, 7, 7, 6, 4, 3, 3, 2, 2, 1)$ and $\beta = (12, 6, 5, 5, 3, 2, 2, 2, 1)$ of 48 and 38, respectively, both have rank vector $[3, -2, 0, 1]$. However, Gupta shows in [1] that for every rank vector, $\mathbf{r}$, there is a *unique* partition π, for which $|\pi|$ is minimum over all partitions with rank vector $\mathbf{r}$. This partition is called the *basis partition* of $\mathbf{r}$. For example, the basis partition of $[3, -2, 0, 1]$ is $(10, 5, 5, 5, 3, 2, 2)$.

Call a partition π of n *basic* if it is the basis partition of its associated rank vector and let $B(n)$ be the set of all basic partitions of n. Of the 42 partitions of 10, only 16 are basic and these are shown in Fig. 1. We consider the empty partition to be a basic partition of $n = 0$.

In the next section, we review some results on basis partitions from [1] and use them to derive a generating function and recurrence for $b(n, d)$, the number of basis partitions of n with Durfee square of size d. In Section 3, we give an alternative characterization of basis partitions, as well as identities describing $B(n)$ in terms of more familiar families of partitions.

2 A Generating Function for Basis Partitions

For a partition π, note that if $d = d(\pi)$, then the Ferrers graph of π (and hence π itself) is completely specified by the first d rows $(\pi_1, \pi_2, \ldots \pi_d)$ and the first d columns $(\pi'_1, \pi'_2, \ldots \pi'_d)$. It will be convenient to view π as a $2 \times d$ array

$$\pi = [\mathbf{x}, \mathbf{y}]_d = \begin{bmatrix} x_1 & \ldots & x_d \\ y_1 & \ldots & y_d \end{bmatrix}$$

where for $1 \le i \le d$, $x_i = \pi_i$ and $y_i = \pi'_i$.

We focus first on the existence and uniqueness of basis partitions.

Theorem 1 (Gupta) *Among all partitions with the same rank vector* $\mathbf{r} = [r_1, \ldots, r_d]$, *there is just one with minimum weight.*

Proof. If $\pi = [\mathbf{x}, \mathbf{y}]_d$ has rank vector $\mathbf{r}$, then since $\mathbf{r} = \mathbf{x} - \mathbf{y}$,

$$|\pi| = \sum_{i=1}^{d}(x_i + y_i) - d^2 = 2\sum_{i=1}^{d} x_i - \sum_{i=1}^{d} r_i \ - \ d^2. \tag{1}$$

The key to minimizing $|\pi|$, then, for fixed $\mathbf{r}$, is to minimize $\sum x_i$. Since $\mathbf{x}, \mathbf{y}$ must satisfy $x_1 \geq x_2 \geq ,\ldots, \geq x_d \geq d$ and $y_1 \geq y_2 \geq, \ldots, \geq y_d \geq d$,

$$x_i \geq \begin{cases} \max(x_{i+1}, x_{i+1} + r_i - r_{i+1}) & \text{if } 1 \leq i < d \\ \max(r_d + d, d) & \text{if } i = d. \end{cases} \tag{2}$$

If $[x_1, \ldots x_d]$ is chosen so that equality holds in (2) and if $y_i = x_i - r_i$, then $\pi = [\mathbf{x}, \mathbf{y}]_d$ actually is a partition, necessarily with rank vector $\mathbf{r}$, and by (1), $|\pi|$ is minimum. Since after minimizing the x_i, the y_i are determined, this π is the unique minimum. □

The following simple test will determine whether a partition is basic.

Lemma 1 *A partition $\pi = [\mathbf{x}, \mathbf{y}]_d$ is basic if and only if both*
(i) $x_d = d$ *or* $y_d = d$ *and*
(ii) *for* $1 \leq i < d$, $(x_i > x_{i+1})$ *implies* $(y_i = y_{i+1})$.

Proof. From the proof of Theorem 1, a partition π is basic if and only if equality holds in (2) and $\mathbf{y} = \mathbf{x} - \mathbf{r}$, that is, if and only if:

$$x_i = \begin{cases} \max(x_{i+1},\ x_{i+1} + (x_i - y_i) - (x_{i+1} - y_{i+1}) & \text{if } 1 \leq i < d \\ \max(x_d - y_d + d,\ d) & \text{if } i = d. \end{cases}$$

For $1 \leq i < d$, note that since π is a partition, $x_i \geq x_{i+1} \geq d$. Thus, equality in this case above occurs if and only if whenever $x_i > x_{i+1}$, we have

$$x_i = x_{i+i} + (x_i - y_i) - (x_{i+1} - y_{i+1}),$$

that is, $y_i = y_{i+1}$. Equality in case $i = d$ occurs if and only if $x_d = d$ or if $x_d = x_d - y_d + d$, that is, $y_d = d$. □

Finally, Gupta [1] notes the following bijection, where $p(n, d)$ denotes the number of partitions of n into at most d parts.

Theorem 2 [Gupta] *Let $\mathbf{r} = [r_1, \ldots r_d]$ and let $\pi = [\mathbf{x}, \mathbf{y}]_d$ be the basis partition of $\mathbf{r}$. The number of partitions of n with rank vector $\mathbf{r}$ is $p(m, d)$ where $m = (n - |\pi|)/2$.*

Proof. If $z_1 \geq z_2 \geq \ldots \geq z_d$ is a partition in the set counted by $p(m,d)$, then $[\mathbf{x}+\mathbf{z}, \mathbf{y}+\mathbf{z}]_d$ is a partition of n with rank vector $\mathbf{r}$. Conversely, if partition $\sigma = [\mathbf{u}, \mathbf{v}]_d$ of n has rank vector $\mathbf{r}$, then by the proof of Theorem 1, for $1 \leq i \leq d$ we have $u_i \geq x_i, v_i \geq y_i$, and $z_i = u_i - x_i = v_i - y_i \geq 0$. If $x_i = x_{i+1}$, then $z_i - z_{i+1} = u_i - u_{i+1} \geq 0$. Otherwise, by Lemma 1 (ii), $y_i \geq y_{i+1}$ and then $z_i - z_{i+1} = v_i - v_{i+1} \geq 0$. Thus, $z_1 \geq z_2 \geq \ldots \geq z_d \geq 0$ and the nonzero terms in this sequence form a partition of $(n - |\pi|)/2$. □

Let $B(n,d)$ be the set of basic partitions of n which have a rank vector of length d, that is, Durfee square of size d, and let $b(n,d) = |B(n,d)|$. The empty partition is the sole element of $B(0,0)$. A partition can be classified according to the length of its rank vector $\mathbf{r}$ and the weight n_0 of the basis partition associated with $\mathbf{r}$. Combining this with Theorem 2 gives the following.

Corollary 1 *The number $p_d(n)$ of partitions of n with Durfee square of size d satisfies*

$$p_d(n) = \sum_{n_0=0}^{n} b(n_0, d) p((n-n_0)/2,\ d),$$

in which it is understood that $p(n,d) = 0$ if n is not an integer.

From this, we can derive the generating function for $b(n,d)$.

Corollary 2 *For $d \geq 0$ the generating function $\Psi_d(q)$ for $b(n,d)$ is*

$$\Psi_d(q) = \sum_{n \geq d^2} b(n,d) q^n = \frac{(1+q)(1+q^2)...(1+q^d)}{(1-q)(1-q^2)...(1-q^d)} q^{d^2}. \tag{3}$$

Proof. Letting $\Phi_k(q)$ denote the well-known generaing function for $p(n,k)$,

$$\Phi_k(q) = \frac{1}{(1-q)\ldots(1-q^k)},$$

we have from Corollary 1 that

$$q^{d^2}[\Phi_d(q)]^2 = \Psi_d(q)\Phi_d(2q).$$

This gives

$$\Psi_d(q) = \frac{(1-q^2)(1-q^4)...(1-q^{2d})}{(1-q)^2(1-q^2)^2...(1-q^d)^2} q^{d^2} = \frac{(1+q)(1+q^2)...(1+q^d)}{(1-q)(1-q^2)...(1-q^d)} q^{d^2}. \tag{4}$$

□

From the partial fraction expansion of (3), for fixed d, we have $b(n,d) \sim 2^d n^{d-1}/(d-1)!$, for large n, compared with $n^{2d-1}/(d!^2(2d-1)!)$, for the number of all partitions of n whose Durfee square has size d.

We can obtain from (3) a recurrence for $b(n,d)$. since

$$\Psi_d(q) = \frac{(1+q^d)}{(1-q^d)} q^{d^2-(d-1)^2} \Psi_{d-1}(q),$$

or equivalently,

$$(1-q^d)\Psi_d(q) = (q^{2d-1} + q^{3d-1})\Psi_{d-1}(q),$$

the following recurrence results from comparing the coefficients of q^n on both sides.

Corollary 3 *We have $b(0,0) = 1$, and $b(n,d) = 0$ if otherwise n or d is nonpositive, and finally, if n and d are both positive, then*

$$b(n,d) = b(n-d,d) + b(n-2d+1,d-1) + b(n-3d+1,d-1).$$

3 An Alternative Characterization of Basis Partitions

Let $D(n,d)$ be the number of partitions of n into distinct parts of size at most d. Since $D(n,d)$ has generating function $(1+q)(1+q^2)\dots(1+q^d)$, Corollary 2 gives:

$$b(n,d) = \sum_{j \geq 0} |D(j,d)| p(n-d^2-j,d). \tag{5}$$

This would suggest that when the $d \times d$ Durfee square is removed from the Ferrers graph of a basis partition, what remains is a partition into distinct parts together with an ordinary partition. However, this is not the case, for example, for $\pi = (7,6,6,4,4,4,4,2,2)$. In this section we present another way to view basis partitions which will make (5) clear.

For a partition $\pi = [\mathbf{x},\mathbf{y}]_d$, let ρ and σ denote the partitions

$$\rho = (x_1 - d, x_2 - d, \dots, x_d - d)$$

and

$$\sigma = (y_1 - d, y_2 - d, \dots, y_d - d),$$

where parts of size 0 are ignored. Then ρ and σ represent the partitions east and south, respectively, of the Durfee square in the Ferrers graph A of π, oriented as shown in Fig. 2. So, π can be represented as the triple $\pi = (d,\rho,\sigma)$.

For convenience below, let $\rho_i = \sigma_i = 0$ if $i > d$.

Theorem 3 *The partition $\pi = (d,\rho,\sigma)$ is basic if and only if the conjugate partitions ρ' and σ' have no common parts.*

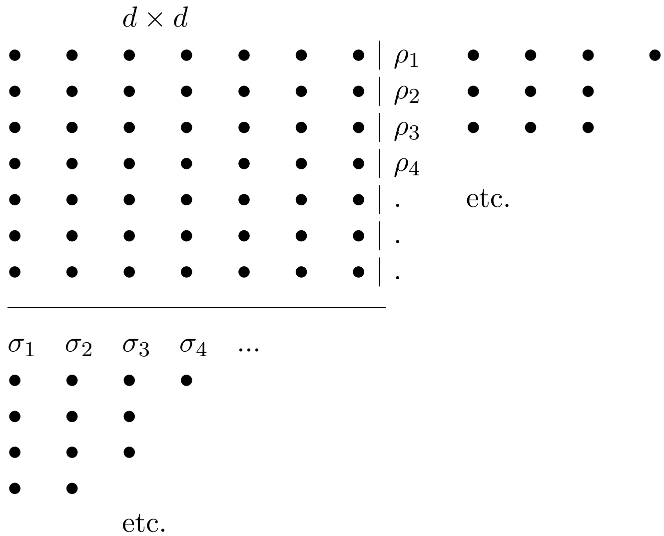

Figure 2: Representation of a partition π with Durfee square of size d as a triple (d, ρ, σ).

Proof. Interpreting Lemma 1 in terms of ρ and σ, $\pi = (d, \rho, \sigma)$ is basic if and only if for $1 \le i \le d$,

$$(\rho_i \;>\; \rho_{i+1}) \quad \Rightarrow \quad (\sigma_i \;= \sigma_{i+1}).$$

We show that $\rho_i > \rho_{i+1}$ if and only if ρ' contains a part of size i (and similarly for σ' if $\sigma_i > \sigma_{i+1}$.) It follows that ρ' and σ' each contain a part of size i if and only if both $\rho_i \;>\; \rho_{i+1}$ and $\sigma_i \;>\; \sigma_{i+1}$, that is, π is not basic.

To complete the proof, if $\rho_i > \rho_{i+1}$, then ρ' contains a part of size i, namely, $\rho'_k = i$ for $k = \rho_i - d$. Conversely, if ρ' contains a part of size i, let j be the position of the last i in ρ'. Then $\rho_i = d + j$ and $\rho_i \;>\; \rho_{i+1}$. $\square$

We now give a bijective proof of (5). Define a mapping

$$\Theta : B(n, d) \longrightarrow \bigcup_{j=0}^{n-d^2} (D(j, d) \times P(n - d^2 - j, d))$$

for $\pi = (d, \rho, \sigma) \in B(n, d)$ by $\Theta((d, \rho, \sigma)) = (\alpha, \beta)$ where (α', β') is obtained from (ρ', σ') by moving all but one copy of each part of ρ' to σ'. For example, $\Theta(12, 9, 6, 6, 3) = \Theta(4, (8, 5, 2, 2), (3)) = ((3, 2, 1, 1), (4, 3, 2, 2, 1, 1, 1))$. Since by Theorem 3, ρ' and σ' had no common part, (d, ρ, σ) can be recovered from (α, β) by moving from α' to β' all parts of α' which occur in β'.

Let $q(n,k,d)$ be the number of partitions of n (into an arbitrary number of parts) that have exactly k distinct parts, and all of those parts are at most d. In view of Theorem 3 we can associate a partition π in $q(n-d^2,k,d)$ with a basic partition (d,ρ,σ) in 2^k ways, according to the number of ways to allocate the k distinct parts of π among ρ' and σ'. Thus we have

$$b(n,d) = \sum_{k\geq 0} q(n-d^2,k,d)2^k. \tag{6}$$

From (6) we have that *the number of basis partitions of* n, $b(n)$, *is even or odd depending, resp., on whether* n *is not or is a square.* Thus, although for most of the famous partition functions of number theory, the discovery of simple parity tests is a very difficult problem, for the number of basis partitions of n it is easy.

We remark further that it is simple to prove this parity result bijectively. By Lemma 1 it follows that the only self-conjugate basis partitions are the ones whose Ferrers graphs are square and that the conjugate of a basis partition is basic. Hence the pairing of each basic partition with its conjugate completes the proof.

References

[1] H. Gupta, The rank-vector of a partition, *Fibonacci Quarterly* **16**, No. 6 (1978) 548-552.

[2] A. O. L. Atkin, A note on ranks and conjugacy of partitions, *Quart. J. Math.* **17**, No. 2 (1966) 335-338.