

MATHEMATICS

ON A CONJECTURE OF RYSER AND MINC

BY

ALBERT NIJENHUIS \*) AND HERBERT S. WILF \*)

(Communicated at the meeting of January 31, 1970)

1. *Introduction*

Let  $A$  be an  $n \times n$  matrix of zeros and ones, and suppose that  $A$  has  $r_i$  ones in its  $i^{\text{th}}$  row ( $i=1, 2, \dots, n$ ). It has been conjectured [2] that for the permanent of  $A$  we have the inequality

$$(1) \quad \text{Per } A < \prod_{i=1}^n (r_i!)^{1/r_i}.$$

The purpose of this paper is to prove

Theorem 1: *There is a universal constant  $\tau = .136708 \dots$  such that*

$$(2) \quad \text{Per } A < \prod_{i=1}^n \{(r_i!)^{1/r_i} + \tau\}.$$

Previous work on inequalities of the form

$$(3) \quad \text{Per } A < \prod_{i=1}^n \varphi(r_i)$$

has been done by MINC [2] who showed that  $\varphi(x) = \frac{1}{2}(x+1)$  is admissible in (3), MINC [3] who found  $\varphi(x) = (1+\sqrt{2})^{-1}(x+\sqrt{2})$  for (3), JURKAT and RYSER [4], WILF [5], and others, but no previous  $\varphi(x)$  has given even the correct first term of the asymptotic behaviour of

$$(4) \quad x^{1/x} = \frac{x}{e} + \frac{\log x}{2e} + \frac{\log \sqrt{2\pi}}{e} + o(1) \quad (x \rightarrow \infty)$$

whereas our theorem 1 above correctly gives the first two terms.

2. *The function  $\varphi(n)$*

Let  $\varphi$  be a fixed function of the positive integers, and suppose (3) holds for all matrices  $A$  of zeros and ones of order  $\leq n-1$ . Now let  $A$  be an  $n \times n$  matrix of this type, with row sums  $r_1, \dots, r_n$ . If any  $r_i = 1$ , then (3) holds for  $A$  provided only that  $\varphi(1) = 1$ . Otherwise, suppose the rows and columns of  $A$  have been permuted, if necessary, so that the

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\*) Research supported in part by the National Science Foundation.

ones in the first column occur in the first  $c$  rows. Expanding by minors down the first column we find

$$\begin{aligned} \text{Per } A &= \sum_{i=1}^c \text{Per } (A^{ii}) \\ &< \sum_{i=1}^c \left\{ \prod_{\substack{k=1 \\ k \neq i}}^c \varphi(r_k - 1) \prod_{k=c+1}^n \varphi(r_k) \right\} \\ &= \sum_{i=1}^c \frac{1}{\varphi(r_i - 1)} \prod_{k=1}^c \varphi(r_k - 1) \prod_{k=c+1}^n \varphi(r_k). \end{aligned}$$

Comparing this expression with the right side of (3) we see that in order to prove (3) for  $A$  it is enough to exhibit a function  $\varphi$  such that

$$(5) \quad \sum_{i=1}^c \frac{1}{\varphi(r_i - 1)} \prod_{k=1}^c \frac{\varphi(r_k - 1)}{\varphi(r_k)} \leq 1$$

for all positive integers  $c, r_1, \dots, r_c$ .

Consider the function  $\varphi$  which is recursively defined by

$$(6) \quad \left\{ \begin{array}{l} \text{(a) } \varphi(1) = 1 \\ \text{(b) } \varphi(n+1) = \varphi(n)e^{1/e\varphi(n)}. \end{array} \right.$$

For this  $\varphi$ , the left side of (5) is

$$\begin{aligned} &\sum_{i=1}^c \frac{1}{\varphi(r_i - 1)} \prod_{k=1}^c e^{-1/e\varphi(r_k - 1)} \\ &= \left\{ \sum_{i=1}^c \frac{1}{\varphi(r_i - 1)} \right\} \exp \left\{ -\frac{1}{e} \sum_{k=1}^c \frac{1}{\varphi(r_k - 1)} \right\} \\ &= xe^{-x/e} \\ &\leq \max_{x \geq 0} xe^{-x/e} \\ &= 1. \end{aligned}$$

It follows that the function  $\varphi$  of (6) is admissible for the inequality (3). The remainder of this paper is devoted to a close study of the recurrence (6) with a view to establishing the relations

$$(7) \quad \varphi(n) = \frac{n}{e} + \frac{\log n}{2e} + \frac{A}{e} + o(1) \quad (n \rightarrow \infty)$$

and

$$(8) \quad \varphi(n) \leq n!^{1/n} + \frac{A - \log \sqrt{2\pi}}{e} \quad (\text{all } n \geq 1).$$

### 3. Asymptotic behaviour of $\varphi$ .

We remark first that putting  $b_n = (e\varphi(n))^{-1}$  in (6) (b) yields

$$b_{n+1} = b_n e^{-b_n} = F(b_n)$$

where  $F(x) = xe^{-x}$ . The asymptotic relation (7) then follows from well-known theorems about the successive iterates of functions  $F$  which have the form

$$F(x) = x - ax^2 + \dots \quad (a > 0)$$

near the origin (see, e.g., [6]). Nonetheless, we prove (7) independently of those results.

First, from (6b),

$$\varphi(n+1) \geq \varphi(n) + \frac{1}{e}$$

and so

$$(a) \quad \varphi(n) \geq \frac{n}{e}.$$

Next, from (6b),

$$\varphi(n+1) \leq \varphi(n) + \frac{1}{e} + \frac{1}{e^2\varphi(n)}$$

which, with (9), yields

$$(10) \quad \varphi(n) \leq \frac{n}{e} + O(\log n) \quad (n \rightarrow \infty).$$

Now write

$$(11) \quad \begin{cases} H(x) = x \left\{ e^{1/x} - 1 - \frac{1}{x} - \frac{1}{2x^2} \right\} \\ = O(x^{-2}) \quad (x \rightarrow 0). \end{cases}$$

Then, with  $y_n = e\varphi(n)$ , we have

$$y_{n+1} = y_n + 1 + \frac{1}{2y_n} + H(y_n).$$

Summing,

$$(11a) \quad \begin{cases} y_n = y_1 + n - 1 + \frac{1}{2} \sum_1^{n-1} \frac{1}{y_k} + \sum_1^{n-1} H(y_k) \\ = n + e - 1 + \frac{1}{2} \sum_1^{n-1} \frac{1}{k} + \frac{1}{2} \sum_1^{n-1} \left( \frac{k - y_k}{ky_k} \right) + \sum_1^{n-1} H(y_k) \\ = n + e - 1 + \frac{1}{2} \log n + \frac{\gamma}{2} + \frac{1}{2} \sum_1^{\infty} \left( \frac{k - y_k}{ky_k} \right) + \sum_1^{\infty} H(y_k) + o(1) \\ = n + \frac{1}{2} \log n + A + o(1) \end{cases}$$

which proves (7), with

$$(11b) \quad A = e - 1 + \frac{\gamma}{2} + \frac{1}{2} \sum_1^{\infty} \left( \frac{k - y_k}{ky_k} \right) + \sum_1^{\infty} H(y_k).$$

## 4. Proof of (8)

We claim that the differences

$$\{\varphi(n) - n!^{1/n}\}_1^\infty$$

increase monotonically. In view of (7) and (4), this will establish (8). We have first,

$$(12) \quad \left\{ \begin{aligned} (n+1)!^{1/n+1} - n!^{1/n} &= \int_n^{n+1} \frac{d}{dx} \{\Gamma(x+1)\}^{1/x} dx \\ &= \int_n^{n+1} \Gamma(x+1)^{1/x} \left\{ \frac{1}{x} \frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{1}{x^2} \log \Gamma(x+1) \right\} dx. \end{aligned} \right.$$

From [1], p. 18, eq. (27),

$$(13) \quad \left\{ \begin{aligned} \frac{\Gamma'(x+1)}{\Gamma(x+1)} &\leq \log(x+1) - \frac{1}{2(x+1)} \\ &\leq \log x + \frac{1}{2x} + \frac{1}{2x(x+1)}. \end{aligned} \right.$$

From [1], p. 22, eq. (9),

$$(14) \quad \left\{ \begin{aligned} \log \Gamma(x+1) &\geq \left(x + \frac{1}{2}\right) \log(x+1) - x - 1 + C \\ &\quad (C = \log \sqrt{2\pi}) \\ &\geq x \log x - x + \frac{1}{2} \log x + C - \frac{1}{4x^2}. \end{aligned} \right.$$

The quantity in braces in (12) is therefore

$$\left\{ \right\} \leq \frac{1}{x} - \frac{\log x}{2x^2} + \frac{\frac{1}{2} - C}{x^2} + \frac{1}{x^3} \quad (x \geq 2).$$

Next, from [1], p. 21, eq. (8),

$$\log \Gamma(x+1) \leq \left(x + \frac{1}{2}\right) \log(x+1) - x - 1 + C + \frac{K}{x}$$

where

$$K = \max_{t>0} \frac{1}{t} \left| \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right|$$

and after some calculation, one finds that

$$(15) \quad \left\{ \begin{aligned} \Gamma(x+1)^{1/x} &\leq \frac{x}{e} + \frac{\log x}{2e} + \frac{C}{e} + \frac{(\log x)^2}{8ex} + \frac{C \log x}{2ex} + \frac{A_2}{x} \\ &\quad (x > x_0) \end{aligned} \right.$$

where  $x_0, A_2$  are any pair such that

$$A_2 \geq \max_{x \geq x_0} \left\{ \frac{A_1}{e} + \frac{C(\log x)^2}{8ex} + \frac{A_1 \log x}{2ex} + \frac{A_1(\log x)^2}{4ex^2} + \frac{(\log x)^3}{8ex} \left[ 1 + \frac{C}{x} + \frac{A_1}{x^2} \right] \right\}$$

and  $A_1 = 2K + 1$ .

Substituting (15), (14) in (12),

$$(16) \quad \left\{ \begin{aligned} (n+1)!^{1/n+1} - n!^{1/n} &\leq \int_n^{n+1} \left( \frac{1}{e} + \frac{1}{2ex} - \frac{(\log x)^2}{8ex^2} \right. \\ &\quad \left. - \frac{(C - \frac{1}{2}) \log x}{2ex^2} + \frac{A_3}{x^2} \right) dx \\ &\quad (n \geq x_0, x_1) \end{aligned} \right.$$

where  $x_1, A_3$  are any pair such that

$$A_3 \geq \max_{x \geq x_1} \left\{ \left( A_2 + \frac{1}{e} \right) + \frac{C(\frac{5}{2} - C)}{2ex} + \frac{(\log x)^2}{8ex^2} + \frac{A_2}{x^2} + \frac{C \log x}{2ex^2} \right\}$$

Then (16) gives

$$(17) \quad \left\{ \begin{aligned} (n+1)!^{1/n+1} - n!^{1/n} &\leq \frac{1}{e} + \frac{1}{2en} - \frac{(\log n)^2}{8en^2} - \left( \frac{C - \frac{1}{2}}{2e} \right) \frac{\log n}{n^2} + \frac{A_4}{n^2} \\ &\quad (n \geq n_2) \end{aligned} \right.$$

where  $n_2, A_4$  are any pair such that  $n \geq n_2$  implies

$$\begin{aligned} & - \frac{[\log(n+1)]^2}{8e(n+1)^2} - \frac{C - \frac{1}{2}}{2e} \frac{\log(n+1)}{(n+1)^2} + \frac{A_3}{n^2} \\ & \leq - \frac{(C - \frac{1}{2}) \log n}{2e} \frac{1}{n^2} + \frac{A_4}{n^2} - \frac{(\log n)^2}{8en^2}. \end{aligned}$$

Now from (6b),

$$\varphi(n+1) \geq \varphi(n) + \frac{1}{e} + \frac{1}{2e^2 \varphi(n)}$$

and since

$$\varphi(n) \leq \frac{n}{e} + \frac{\log n}{2e} + A_5$$

we find that

$$(18) \quad \varphi(n+1) - \varphi(n) \geq \frac{1}{e} + \frac{1}{2ne} + \frac{\log n}{4en^2} - \frac{A_5}{2n^2}$$

Subtracting (17) from (18),

$$\begin{aligned} & [\varphi(n+1) - (n+1)!^{1/n+1}] - [\varphi(n) - n!^{1/n}] \\ & \geq \frac{(\log n)^2}{8en^2} - \frac{2C-1}{4e} \frac{\log n}{n^2} - \frac{A_5 + 2A_4}{n^2} \\ & \geq 0 \end{aligned}$$

for  $n \geq n^*$ . By simple estimations, one can take, successively,  $K = 1/12$ ,  $(x_0, A_2) = (1, 1.5)$ ,  $(x_1, A_3) = (1, 1.9)$ ,  $(n_2, A_4) = (1, 2)$ ,  $A_5 = 1$ , and finally  $n^* = e^{12} < 5000$ , which proves the monotonicity of the sequence for  $n \geq 5000$ . By actual calculation<sup>1)</sup> one observes the monotonicity for  $n \leq 5000$  also, completing the proof.

### 5. Conclusion

Since  $\{\varphi(n) - n!^{1/n}\} \uparrow$  we have from (7) and (4) the desired inequality (8). By computation we found, for example

$$\begin{aligned} \varphi(99,000) &= 36,422.65517926 \dots \\ (99,000)!^{1/99,000} &= 36,422.5186529 \dots \end{aligned}$$

We remark that the constant  $A$  can be exhibited as the solution of a certain functional equation, as is characteristic of problems of functional iteration. Suppose we let  $\varphi(n, \xi)$  denote the solution of the recurrence (6) (b) with the starting value  $\varphi(1, \xi) = \xi$ , in place of (6) (a). One has again the formula (7),

$$\varphi(n, \xi) = \frac{n}{e} + \frac{\log n}{2e} + \frac{A(\xi)}{e} + o(1) \quad (n \rightarrow \infty).$$

Now, since

$$\varphi(n+1, \xi) = \varphi(n, \xi e^{1/e^{\xi}})$$

there follows

$$(19) \quad A(\xi e^{1/e^{\xi}}) = 1 + A(\xi).$$

Thus, our constant  $A$  is  $A(1)$ , where  $A(\xi)$  is a solution of (19). We remark finally that the method will not prove the conjecture in the sense that the function  $\varphi(x) = x!^{1/x}$  does not satisfy the relation (5), for example when  $r_1 = \dots = r_m = 2m$ ;  $r_{m+1} = \dots = r_{m+n} = 2n$ .

The constant  $A = 1.2905502 \dots$  was computed from (11b) in which the series shown are rapidly convergent.

*University of Pennsylvania*

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<sup>1)</sup> We gratefully acknowledge the assistance of the University of Pennsylvania Computer Center in providing the time for numerous calculations related to this paper.

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