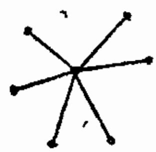
THE EIGENVALUES OF A GRAPH AND ITS CHROMATIC NUMBER

H. S. WILF

Let G be a finite, connected, undirected graph, without loops or multiple edges. If v is a vertex of G, the degree of v, $\rho(v)$, is the number of edges emanating from v. R. L. Brooks has shown [1] that

$$k \leqslant 1 + \max_{v \in G} \rho(v) \tag{1}$$

where k is the chromatic number of G, with equality if and only if G is a complete graph or an odd circuit. The estimator (1) may be crude if G has just a few vertices of high degree. An extreme case is the star graph on n vertices



(2)

for which $k = 2$ and (1) gives only $k \leqslant n$. It seems, therefore, desirable to find an upper estimate, of the character of (1), which is more global in nature, and therefore is less sensitive to the idiosyncrasies of a few uninfluential vertices.

With G we associate the $n \times n$ vertex-adjacency matrix $A = A[G]$, whose i, j entry is 1 if vertices i and j are connected and 0 otherwise. Let $\lambda = \lambda[G] = \lambda_{\max}(A)$ denote the largest eigenvalue of A.

THEOREM. *We have*

$$k \leqslant 1 + \lambda \tag{3}$$

with equality if and only if G is a complete graph or an odd circuit.

Remark. By the Perron–Frobenius theorem, $\lambda \leqslant \max_{v \in G} \rho(v)$, always, so (3) is never inferior to (1). For the graph (2), (3) gives $k = O(\sqrt{n})$.

Proof of the theorem. Let the chromatic number of G be k. It may be that we can remove a vertex and all edges incident to that vertex from G without lowering the chromatic number. We do this repeatedly, if possible, until a *critical graph* [2] results, *i.e.*, a graph such that the removal of any star lowers the chromatic number. Let this critical graph be called G_c, and suppose it has $m \leqslant n$ vertices. Consider the following three matrices: $A[G_c]$, the $m \times m$ adjacency matrix of G_c; A', the $n \times n$ matrix obtained

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from $A[G]$ by replacing the deleted-vertex rows and columns by zeros; $A[G]$ itself. We have

$$\lambda[G_c] = \lambda_{\max}(A') \leqslant \lambda[G] \equiv \lambda \tag{4}$$

the first equality being obvious, and the inequality following from the entry-by-entry domination of $A[G]$ over A'.

On the other hand, it is well-known (and indeed, clear) that in a k-chromatic *critical* graph the degree of each vertex is at least $k-1$, and by well-known results about matrices with non-negative elements,

$$\lambda[G_c] \geqslant k-1 \tag{5}$$

since the smallest row sum in $A[G_c]$ is $\geqslant k-1$, proving (3).

Now suppose $k = 1+\lambda$. Then equality holds in (5), so all the row sums of $G[G_c]$ are equal to $k-1$. Suppose $k > 2$. By the theorem of Brooks referred to above, we have equality in (1), and so G_c is a complete graph on k vertices. Hence, after renumbering the vertices, if necessary, $A[G]$ can be brought into the form of an $n \times n$ matrix whose upper left $k \times k$ block is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \\ . & . & . & . & \dots & . \\ . & . & . & . & \dots & . \\ . & . & . & . & \dots & . \\ 1 & 1 & 1 & 1 & \dots & 0 \end{pmatrix}$$

Consider the n-vector $x = (1, 1, \dots, 1, \epsilon, 0, 0, \dots, 0)$ whose $(k-1)$-st component is $\epsilon > 0$. Then

$$\lambda \geqslant \frac{(x, A[G]x)}{(x, x)} \geqslant \frac{k(k-1) - 2\epsilon \sum_{j=1}^{k} a_{j,k+1} + O(\epsilon^2)}{k - \epsilon^2}$$

which is $> k-1$, a contradiction, unless $a_{j,k+1} = 0$ $(j = 1, \dots, k)$. Moving the ϵ to a different position in x, we conclude that $a_{jr} = 0$ $(j = 1, \dots, k;$ $r = k+1, \dots, n)$, hence G is disconnected, a contradiction, and so $n = k$. The case $k = 2$ can be handled similarly.

COROLLARY. *Let G have E edges and n vertices. Then*

$$k \leqslant \left\{2\left(1 - \frac{1}{n}\right) E\right\}^{\frac{1}{2}} + 1$$

with equality only for complete graphs.

Proof. If $\Sigma\lambda_i = 0$, then

$$\max_i \lambda_i \leqslant \left\{\left(1 - \frac{1}{n}\right) \sum_{i=1}^{n} \lambda_i^2\right\}^{\frac{1}{2}}$$

hence

$$\lambda_{\max}(A) \leqslant \left\{\left(1-\frac{1}{n}\right)\operatorname{Trace}(A^2)\right\}^{\frac{1}{2}}$$

$$=\left\{2\left(1-\frac{1}{n}\right)E\right\}^{\frac{1}{2}}.$$

References

1. R. L. Brooks, " On coloring the nodes of a network ", *Proc. Cambridge Phil. Soc.*, 37 (1941), 194–197.

2. G. A. Dirac, " Note on the colouring of graphs ", *Math. Zeitschrift*, 54 (1951), 347–353 .

Department of Mathematics,
University of Pennsylvania,
Philadelphia 19104.