

# A Note on an Identity of Feigin-Stoyanovsky

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We use the standard notation of hypergeometric series by defining

$$(a; q)_k = \begin{cases} ((1 - aq^{-1})(1 - aq^{-2}) \dots (1 - aq^k))^{-1}, & \text{if } k < 0; \\ 1, & \text{if } k = 0; \\ (1 - a)(1 - aq) \dots (1 - aq^{k-1}), & \text{if } k > 0. \end{cases}$$

Then the identity of the title, which was communicated to me by Alexandre Kirillov, is

$$\sum_{k=0}^m \frac{(q^{k+1}; q)_{-k} (q^{m-k+1}; q)_{k-m} (bq^{k+1}; q)_{k-m} (q^{m-k+1}/b; q)_{-k}}{(q^{m+1}; q)_{-m} (abq; q)_{k-m} (cq; q)_{k-m} (cq/b; q)_{-k}} = \frac{1}{(acq; q)_{-m}}. \quad (1)$$

## 1 The computer proof via the $q$ -WZ method

Following the WZ paradigm, we define  $F(m, k)$  to be the summand in (1) divided by the right hand side, i.e.,

$$F(m, k) = \frac{(q^{k+1}; q)_{-k} (q^{m-k+1}; q)_{k-m} (bq^{k+1}; q)_{k-m} (q^{m-k+1}/b; q)_{-k} (acq; q)_{-m}}{(q^{m+1}; q)_{-m} (abq; q)_{k-m} (cq; q)_{k-m} (cq/b; q)_{-k}}, \quad (2)$$

and now what we have to prove is that

$$f(m) \stackrel{\text{def}}{=} \sum_k F(m, k) = 1 \quad (m \geq 0). \quad (3)$$

In order to do this it will suffice to exhibit a function  $G(m, k)$  such that

$$F(m, k) - F(m-1, k) = G(m, k) - G(m, k-1), \quad (4)$$

and such that  $G(m, k)$  vanishes, for each fixed  $m$ , as  $k \rightarrow \pm\infty$ .

For suppose we have found such a  $G$ . Then sum both sides of (4) over all integers  $k$ , observe that the right side telescopes to zero, and obtain the result that

$$f(m) - f(m-1) = 0 \quad (m \geq 0),$$

where  $f$  is defined in (3) above. But this clearly implies that  $f$  is constant on the nonnegative integers, and since  $f(0) = 1$ , we will have shown the desired result (3).

A function  $G$  which satisfies the above properties is given by

$$G(m, k) = R(q^m, q^k)F(m, k) \quad (5)$$

where  $F(m, k)$  is given by (2) and the rational function  $R$  is

$$R(x, y) = \frac{qx(y-a)(by-c)(y-x)(by-x)}{(x-by^2)(x-1)(x-abqy)(x-cqy)}. \quad (6)$$

To prove that the WZ equation (4) is satisfied, one can divide it by  $F(m, k)$ , and show that

$$1 - \frac{F(m-1, k)}{F(m, k)} = R(q^m, q^k) - R(q^m, q^{k-1}) \frac{F(m, k-1)}{F(m, k)}. \quad (7)$$

Both sides of this equation are *rational functions* of  $x = q^m, y = q^k$ , so the verification is straightforward, by clearing of fractions and checking the resulting polynomial identity.

To make this note self-contained, so the reader can readily check the entire computation we will exhibit the left and right ratios  $F(m-1, k)/F(m, k)$  and  $F(m, k-1)/F(m, k)$ . These are obtained directly from the definition (2) of  $F$ , and they are

$$\frac{F(m-1, k)}{F(m, k)} = \frac{(x-acq)(y-x)(by-x)(x-bqy^2)}{(1-x)(x-abqy)(x-cqy)(by^2-x)} \quad (x = q^m; y = q^k) \quad (8)$$

$$\frac{F(m, k-1)}{F(m, k)} = \frac{q(1-y)(1-by)(x-abqy)(x-cy)(by^2-qx)(by^2-q^2x)}{(y-aq)(y-qx)(by-cq)(by-qx)(x-by^2)(qx-by^2)} \quad (x = q^m; y = q^k) \quad (9)$$

A complete verification of this proof now proceeds as follows.

1. Check that the left and right ratios of the summand  $F$  are as shown above in (8) and (9).
2. Substitute these in (7), using the definition (6) of the rational function  $R$ , and thereby verify that (7) is correct.
3. The truth of the WZ equation (4) is now established. The remainder of the argument is as given previously.

## 2 Comments

The computerized portion of the proof takes as input the summand  $F(m, k)$  of (3), and finds the rational function  $R$ , of (6), such that if  $G$  is defined by (5), then the WZ equation (4) is satisfied. There are several software implementations of the  $q$ -Zeilberger algorithm that can routinely accomplish this task. The one we used is the one due to Axel Riese, of RISC-Linz, and which can be found at

< <http://www.risc.uni-linz.ac.at/research/combinat/risc/software/qZeil/> >.