

Note

A Method and Two Algorithms on the Theory of Partitions

A. NIJENHUIS AND H. S. WILF*

*Department of Mathematics, University of Pennsylvania,
Philadelphia, Pennsylvania 19174*

Communicated by Mark Kac

Received October 31, 1973

In this note we describe a general principle for selecting at random from a collection of combinatorial objects, where "at random" means in such a way that each of the objects has equal probability, *a priori*, of being selected. We apply this principle by displaying two algorithms, the first of which will select a random partition of an integer n , and the second will choose a random partition of a set S of n elements.

We begin by stating the general principle, albeit rather vaguely. Let a_n be the number of combinatorial objects of order n . Suppose the numbers a_n satisfy a recurrence relation of the form

$$a_n = \sum_{m < n} \alpha_{m,n} a_m, \tag{1}$$

in which $\alpha_{m,n} \geq 0$. Suppose further that the recurrence (1) can be given a combinatorial interpretation, by which we mean that there is a proof of (1) in which, by an explicit construction, the a_m objects of order m are extended to $\alpha_{m,n} a_m$ objects of order n . Then we have an algorithm for selecting an object of order n at random: first choose a value of m , $0 \leq m \leq n - 1$ according to the probabilities

$$\text{Prob}(m) = \frac{\alpha_{m,n} a_m}{a_n} \quad (m = 0, 1, \dots, n - 1). \tag{2}$$

Then, inductively, select a random object of order m , and extend it as described in the proof.

We make the above statement precise in two cases. First, if $p(n)$ is the

* Research carried out under John Simon Guggenheim Memorial Fellowship.

number of partitions of the integer n , then an identity of Euler asserts that

$$np(n) = \sum_{m < n} \sigma(n - m) p(m) \quad (p(0) = 1), \quad (3)$$

where $\sigma(m)$ is the sum of the divisors of the integer m , while if a_n is the number of partitions of a *set* of n elements, then the a_n satisfy

$$a_n = \sum_{m=0}^{n-1} \binom{n-1}{m} a_m \quad (a_0 = 1). \quad (4)$$

Next we must ask for the combinatorial interpretations of (3), (4). For (3), let π denote a fixed partition of an integer $m < n$. Let d be a divisor of $n - m$. With the pair (π, d) we associate exactly d copies of a single partition π' , of n . Here π' is obtained by adjoining to the partition π , of m , exactly $(n - m)/d$ copies of d .

We claim that as d varies over all divisors of $n - m$ and π varies over all partitions of m ($m = 0, 1, \dots, n - 1$), each partition of n is constructed exactly n times by this process, which will establish (3). Indeed, let

$$\pi' : n = \mu_1 r_1 + \mu_2 r_2 + \dots + \mu_k r_k \quad (5)$$

be a fixed partition of n , where the r_i are the *distinct* parts of π' and the μ_i are their multiplicities. Then π' is constructed by adjoining t copies of r_i to a partition of $n - tr_i$ for each $1 \leq t \leq \mu_i$, $1 \leq i \leq k$, and by replicating the resulting partition of n r_i times, which gives a total of

$$\sum_{i=1}^k r_i \sum_{t=1}^{\mu_i} 1 = \mu_1 r_1 + \dots + \mu_k r_k = n,$$

times altogether, as claimed.

As for partitions of sets, the interpretation of (4) is well known, but we include it for completeness: If π is a partition of $\{1, 2, \dots, m\}$, let S be a subset $\{s_1, \dots, s_m\}$ of m elements chosen from $\{1, 2, \dots, n - 1\}$. With (π, S) we associate the partition of $\{1, 2, \dots, n\}$ in which elements s_i and s_j from S are in the same class iff their subscripts i and j are in the same class of π , while all elements not in S are in a single class. It is easy to check that each partition of $\{1, 2, \dots, n\}$ occurs exactly once. (We have assumed $s_1 < s_2 < \dots < s_m$.)

We can now describe the algorithms.

ALGORITHM I. *Given n , select a random partition of n .*

- (A) Set $n' \leftarrow n$, $P \leftarrow$ empty partition.
- (B) Choose an integer $m < n'$ according to the probabilities

$$\text{Prob}(m) = \frac{\sigma(n' - m) p(m)}{n' p(n')} \quad (m = 0, 1, \dots, n' - 1).$$

- (C) Choose a divisor d of $n' - m$ according to the probabilities

$$\text{Prob}(d) = d/\sigma(n' - m) \quad (d \mid (n' - m)).$$

- (D) Adjoin to the partition P $(n' - m)/d$ copies of d .
- (E) Replace n' by m .
- (F) If $n' = 0$, stop. Otherwise return to step (B).

ALGORITHM S. Given n , select a random partition of $U = \{1, 2, \dots, n\}$.

- (A) Set $U' \leftarrow U$, $m \leftarrow n$, $P \leftarrow$ unique partition of empty set.
- (B) Choose an integer $k < m$ according to the probabilities

$$\text{Prob}(k) = \binom{m-1}{k} \frac{a_k}{a_n} \quad (0 \leq k \leq m-1).$$

- (C) Let l be the largest element of U' . Choose a random k -subset S of $U' - \{l\}$.

- (D) Adjoin $U' - S$ as a single class to P . Stop if $k = 0$. Otherwise set $m \leftarrow k$, $U' \leftarrow S$ and return to step (B).

It remains to show that all partitions have equal probability ($= p(n)^{-1}$ or a_n^{-1} , respectively). We show this in the case of partitions of an integer, the other case being similar.

Let

$$\pi' : n = \mu_1 r_1 + \dots + \mu_k r_k \tag{6}$$

be a fixed partition of n . Inductively, suppose that for all $n' < n$ it has been shown that our algorithm produces partitions of n' with all equal probabilities. Then $\text{Prob}(\pi')$ is a sum over all partitions π'' of integers $m < n$ of $\text{Prob}(\pi'')$ multiplied by the probability that on the next step π'' is extended to π' :

$$\begin{aligned} \text{Prob}(\pi') &= \sum \text{Prob}(\pi'') \text{Prob}(\pi''_m \rightarrow \pi') \\ &= \sum 1/p(m) \text{Prob}(\pi''_m \rightarrow \pi'). \end{aligned} \tag{7}$$

The last factor vanishes unless π' can result from π''_m by adjunction of

exactly t copies of one part r_i . In the latter case, if π' is the partition (6) then

$$\begin{aligned} \text{Prob}(\pi_m'' \rightarrow \pi') &= \text{Prob}(m = n - tr_i) \text{Prob}(d = r_i) \\ &= \left\{ \frac{\sigma(tr_i) p(m)}{np(n)} \right\} \left\{ \frac{r_i}{\sigma(tr_i)} \right\} \\ &= \frac{p(m)}{np(n)} r_i. \end{aligned}$$

The sum (7) over all candidates π_m'' for extension to π' in a single step becomes just

$$\begin{aligned} \text{Prob}(\pi') &= \sum_{i=1}^k \sum_{t=1}^{\mu_i} \frac{1}{p(m)} \frac{p(m)}{n p(n)} r_i \\ &= \frac{1}{n p(n)} \sum_{i=1}^k \mu_i r_i \\ &= \frac{1}{p(n)}, \end{aligned}$$

as required.

ACKNOWLEDGMENTS

We express our indebtedness to Professor Basil Gordon and Professor Marcel Schützenberger for stimulating conversations relating to this work.

Note added in proof. A generalization of these ideas, along with other applications, appears in [1].

REFERENCE

1. A. NIJENHUIS AND H. S. WILF, "Combinatorial Algorithms," Academic Press, New York, 1975.