Some unsolved problems

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Here are some mathematical problems that are, as far as I know, unsolved, and which I have encountered in recent work.

1 Series for π

A great many rapidly converging series for π are known. Most often they are of the form

$$\pi = \sum_{n\geq 0} t_n,$$

where t_n is a hypergeometric term, that is, t_{n+1}/t_n is a rational function of n. Our question roughly is this - how fast can such a series converge to π?

Of course without further conditions the question is trivial, so we must add that the hypergeometric term t_n has rational coefficients. The known series all seem to converge exponentially fast, that is, $C = \lim t_n^{1/n}$ exists and is finite and nonzero. Given any such series it is simple to construct another one in which C is replaced by C^2, so there exist such representations of π in which the constant C is arbitrarily small. But can it be 0? That is, can such a series converge superexponentially fast?

We ask the question precisely as follows.

Does there exist an entire function $f(z) = \sum_{n\geq 0} a_n z^n$ such that the coefficients a_n are hypergeometric terms over the rational numbers, and $f(1) = \pi$?

It is worth noting that if we replace "π" by "e" in the above it becomes quite trivial, since e^z is an entire function.

2 Growth of partition functions

Let S be a set of positive integers and let M be a set of nonnegative integers containing 0. Let $p(S, M; n)$ denote the number of partitions of n whose parts all lie in S and the multiplicities of whose parts all lie in M. The following question was encountered in some joint work with Rod Canfield.

Can we choose S and M so that

1. *S and M are both infinite sets, and*
2. *all sufficiently large integers n are represented, i.e., $p(S, M; n) > 0$ for all large n, and*

3. $p(S,M;n)$ is of at most polynomial growth in n?

3 A problem in asymptotics

Let

$$f(z) = \frac{\arctan\sqrt{2e^{-z}-1}}{\sqrt{2e^{-z}-1}}.$$

If $f(z) = \sum_{n\geq 0} a_n z^n$, find the first term of the asymptotic behavior of the a_n's.

4 The quadratic character of binomial coefficients

An integer n is p-balanced if, among the nonzero values of $\{\binom{n}{k}\}_{k=0}^{n}$ (mod p) there are equal numbers of quadratic residues and quadratic nonresidues mod p. Say that the prime p is special if no integer n, $0 \leq n < p$ is p-balanced. The primes 2, 3, 11 are special, and no other primes $< 1,000,000$ are special.Are there any more special primes? See my paper with Richard Garfield, *The distribution of the binomial coefficients modulo p*, J. Number Theory **41** (1992), 1-5, which is available from my web site.

5 Young tableaux

In 1992 I found[1] a relationship between the numbers of standard Young tableaux of n cells whose first row has length $\leq k$, on the one hand, and the number of permutations of n letters whose longest ascending subsequence has length $\leq k$, on the other.

More precisely, let k be an even number, let $y_k(n)$ be the number of Young tableaux of n cells whose first row has length $\leq k$, and let $u_k(n)$ be the number of permutations of n letters that have no ascending subsequence of length $> k$. Then it is true that

$$\binom{2n}{n} u_k(n) = \sum_r \binom{2n}{r} (-1)^r y_k(r) y_k(2n-r).$$

I discovered this relationship by using analytical methods, but the relationship itself betrays none of its analytical origins, and is in fact a purely combinatorial, finite relation. That being the case, it should have a purely combinatorial derivation. Find one.

6 Distinct multiplicities

Let $T(n)$ be the set of partitions of n for which the (nonzero) multiplicities of its parts are all different, and write $f(n) = |T(n)|$. See Sloane's sequence `A098859` for a table of values. Find any interesting theorems about $f(n)$. The mapping that sends a partition of n to another partition of n in which the roles of parts

[1] Ascending subsequences of permutations and the shapes of Young tableaux, J. Combinatorial Theory, Ser. A **60** (1992), 155-157.

and multiplicities are interchanged is a well defined involution on $T(n)$, which is how I arrived at the study of this problem.

7 Toeplitz determinants

Find $f(n)$, the number of monomials in the expansion of the $n \times n$ general Toeplitz determinant

$$\det\left(\left(a_{|i-j|}\right)_{i,j=1,\ldots,n}\right).$$

See Sloane's sequence `A019447` for the values up to $n = 11$. Is $f(n)$ of superexponential growth?

8 Chromatic number

By a theorem of the late George Szekeres and myself[2], we have

$$\chi(G) \leq 1 + \max_{G' \subseteq G} \delta(G'),$$

where χ is the chromatic number and δ is the minimum degree of the vertices. For precisely which graphs G does the sign of equality hold?

[2] An inequality for the chromatic number of a graph, (with G. Szekeres), *J.* Combinatorial Theory, **4** (1968), 1-3.