On the outstanding elements of permutations

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To Dominique Foata, on his sixtieth birthday

If σ is a permutation of n letters, say that j is an *outstanding element* (Fr. *élément saillant*) of σ if whenever $i < j$ we have $\sigma(i) < \sigma(j)$. A famous theorem of A. Rényi [R] (see also [C]) states that the number of permutations of n letters that have k outstanding elements is equal to the number that have k cycles, and therefore there are $\left[{n \atop k}\right]$ such permutations, the latter being the unsigned Stirling number of the first kind.

Here we investigate other properties of the outstanding elements, and show the following:

(I) The *average* of the rth outstanding elements, over all permutations of n letters that have that many, is $\sim (\log n)^{r-1}/(r-1)!$, for fixed r and $n \to \infty$.

(II) The *average value of a permutation* σ at its rth outstanding element, among all permutations that have that many, is

$$\sim (1 - \frac{1}{2^r})n \qquad (n \to \infty).$$

(III) Let $1 \le j_1 < j_2 < j_3 < \cdots < j_m \le n$ be fixed integers, and suppose that we attach a symbol $s(j) =$ 'Y' or 'N' to each of these j's. Then the probability that a permutation of n letters *does* have an outstanding element at each of the j_ν that is marked 'Y', and *does not* have an outstanding element at any of those that are marked 'N', is

$$\prod_{s(j_\nu)=\text{`N'}} (1 - \frac{1}{j_\nu}) \prod_{s(j_\nu)=\text{`Y'}} \frac{1}{j_\nu}.$$

(IV) Let $S \subseteq \{1, 2, \ldots, n\}$ with $1 \in S$. The probability that a permutation of n letters has exactly the set S for its set of outstanding elements is

$$\frac{1}{n} \prod_{j \in S \setminus \{1\}} \frac{1}{(j-1)}.$$

(V) Let ϕ be a function from the positive integers to, say, the complex numbers. Associate with each permutation σ the statistic $\chi(\sigma) = \sum_j' \phi(j)$, in which the sum runs over all outstanding elements j of σ. Then the average of $\chi(\sigma)$, over all permutations σ of n letters, is

$$\overline{\chi}_n = \sum_{j=1}^{n} \frac{\phi(j)}{j}.$$

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In particular, with the customary abuse of language we can say that *the average of the sum of the reciprocals of the outstanding elements of a random permutation is* $\pi^2/6$.

(VI) For the grand generating function

$$f_n(x_1,\ldots,x_n)=\sum_{\sigma\in S_n}\Bigl(\prod_{i\,\in\,\mathrm{outs}(\sigma)}x_i\Bigr),$$

in which the sum is over all n-permutations σ, and the product is over all of the outstanding elements i of σ, we have

$$f_n(x_1,\ldots,x_n)=\prod_{j=1}^{n}(x_j+j-1).$$

Proof of (I)

Let $f_n(r,s,t)$ be the number of permutations σ of n letters with $\ge r$ outstanding elements, the rth of which is s, with value $\sigma(s)=t$. That is, f_n is the number of σ such that $\sigma(s)=t$, and s is the rth integer i such that $\sigma(i)$ exceeds all $\sigma(j)$ for $j<i$.

We claim that

$$f_n(r,s,t)=\binom{t-1}{s-1}\begin{bmatrix}s-1\\r-1\end{bmatrix}(n-s)!.\qquad(0\le s,t\le n)$$

Indeed, for the first $s-1$ values of σ we can choose any $s-1$ of the values $\le t-1$. These $s-1$ values can be arranged in any order that corresponds to a permutation of $s-1$ letters that has exactly $r-1$ outstanding elements, and there are exactly $\left[\begin{smallmatrix}s-1\\r-1\end{smallmatrix}\right]$ such permutations. Finally the last $n-s$ values of σ can be arranged in any order.

We now ask for the *average* of the rth outstanding element of a permutation of n letters, among those that have at least r such elements. If we denote this by $\phi_n(r)$ then we have

$$\begin{aligned}\phi_n(r)&=\frac{\sum_{s,t=1}^{n}sf_n(r,s,t)}{\sum_{s,t=1}^{n}f_n(r,s,t)}\\&=\frac{\sum_{s=1}^{n}s\frac{1}{s!}\left[\begin{smallmatrix}s-1\\r-1\end{smallmatrix}\right]}{\sum_{s=1}^{n}\frac{1}{s!}\left[\begin{smallmatrix}s-1\\r-1\end{smallmatrix}\right]}.\end{aligned}$$

Interestingly, the sum in the numerator simplifies to

$$\phi_n(r)=\frac{1}{(n-1)!}\begin{bmatrix}n\\r\end{bmatrix}.$$

Indeed, to establish that

$$\sum_{s=1}^{n}\frac{1}{(s-1)!}\begin{bmatrix}s-1\\r-1\end{bmatrix}=\frac{1}{(n-1)!}\begin{bmatrix}n\\r\end{bmatrix},$$

just take the usual recurrence relation for the Stirling number version of the Pascal triangle, viz.

$$\begin{bmatrix} s \\ r \end{bmatrix} = \begin{bmatrix} s-1 \\ r-1 \end{bmatrix} + (s-1)\begin{bmatrix} s-1 \\ r \end{bmatrix},$$

divide by $(s-1)!$, and sum from $s = 1$ to n.

Therefore

$$\begin{aligned} \phi_n(r) &= \frac{\frac{1}{(n-1)!}\left[{n \atop r}\right]}{\sum_{s=1}^{n} \frac{1}{s!}\left[{s-1 \atop r-1}\right]} \\ &= \frac{\frac{1}{(n-1)!}\left[{n \atop r}\right]}{1 - \sum_{s>n} \frac{1}{s!}\left[{s-1 \atop r-1}\right]}. \end{aligned} \tag{1}$$

Asymptotically, for fixed r, as $n \to \infty$, it is well known that

$$\frac{1}{(n-1)!}\begin{bmatrix} n \\ r \end{bmatrix} \sim \frac{(\log n)^{r-1}}{(r-1)!},$$

and indeed the complete asymptotic expansion can be found in [W]. Hence the sum that appears in the denominator of the last member of (1) is

$$\sim \frac{1}{(r-2)!}\sum_{s>n} \frac{(\log s)^{r-2}}{s(s-1)} = o(1) \qquad (n \to \infty).$$

This completes the proof of assertion (I).

Proof of (II)

Next we consider the average size of $\sigma(j)$, where j is the rth outstanding element of a permutation of n letters.

Now $\sum_s f_n(r, s, t)$ is the number of n-permutations σ that have at least r outstanding elements, at the rth of which the value of σ is t. Hence for fixed r the average value of σ at its rth outstanding element is

$$\begin{aligned} \frac{\sum_{s,t\le n} t f_n(r,s,t)}{\sum_{s,t\le n} f_n(r,s,t)} &= \frac{1}{\left(\left[{n \atop r}\right] + \left[{n \atop r+1}\right] + \cdots\right)} \sum_{s,t\le n} t f_n(r,s,t) \\ &\sim \frac{1}{n!}\sum_{s,t\le n} t f_n(r,s,t) \\ &= \frac{1}{n!}\sum_{s=r}^{n} \begin{bmatrix} s-1 \\ r-1 \end{bmatrix} (n-s)!s \sum_{t=s}^{n} \binom{t}{s} \\ &= (n+1)\sum_{s=r}^{n} \begin{bmatrix} s-1 \\ r-1 \end{bmatrix} \frac{s}{(s+1)!} \\ &\sim n\sum_{s=r}^{\infty} \frac{1}{(s-1)!(s+1)} \begin{bmatrix} s-1 \\ r-1 \end{bmatrix} \\ &\overset{\text{def}}{=} c_r n, \end{aligned}$$

say.

To evaluate the $\{c_r\}$ in closed form we can multiply the generating function

$$(1-t)^{-x} = \sum_{n,k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix} x^k \frac{t^n}{n!}$$

by tdt and integrate from $t=0$ to 1, obtaining

$$\sum_{k\geq 0} c_{k+1} x^k = \frac{1}{1-x} - \frac{1}{2-x},$$

whence $c_r = 1-2^{-r}$ for $r \geq 1$, as required. A closer analysis shows that in fact the average is $(1-2^{-r})n + O((\log n)^{r-2})$.

Proofs of (III), (IV)

The result (III) is a generalization of a theorem of R. V. Kadison [K], who discovered the case where the m labels are 'NN$\cdots$NY', and proved it by the sieve method.

Our proof of (III) is by induction on n. Suppose it has been proved that for all $1 \leq j_1 < \ldots < j_m \leq n-1$, and for all sets $\mathcal{L}$ of m labels 'Y' or 'N' that the number of permutations of $n-1$ letters that are as described in (III) above is

$$f_{n-1}(\mathbf{j}, \mathcal{L}) = (n-1)! \prod_{s(j_\nu)=\text{'N'}} (1 - \frac{1}{j_\nu}) \prod_{s(j_\nu)=\text{'Y'}} \frac{1}{j_\nu}.$$

Now suppose we are given a set $\mathbf{j}$ of j's and a set $\mathcal{L}$ of labels for n-permutations. Suppose first that $j_m \leq n-1$. Then the number of matching permutations is clearly nf_{n-1}, and we are done. Suppose next that $j_m = n$ and that it is labeled 'Y'. Then every such permutation σ must have $\sigma(n) = n$, hence their number is (the primes denote deletion of the last entry)

$$\begin{aligned} f_n(\mathbf{j}, \mathcal{L}) &= f_{n-1}(\mathbf{j}', \mathcal{L}') \\ &= (n-1)! \prod_{\substack{s(j_\nu)=\text{'N'} \\ \nu<m}} (1 - \frac{1}{j_\nu}) \prod_{\substack{s(j_\nu)=\text{'Y'} \\ \nu<m}} \frac{1}{j_\nu} \\ &= n! \prod_{\substack{s(j_\nu)=\text{'N'} \\ \nu\leq m}} (1 - \frac{1}{j_\nu}) \prod_{\substack{s(j_\nu)=\text{'Y'} \\ \nu\leq m}} \frac{1}{j_\nu}, \end{aligned}$$

as required. The last case, in which $j_m = n$ and it is labelled 'N', is virtually identical, and is omitted. ∎

The result (IV) above is the special case of (III) in which $m = n$ and $j_1 = 1, j_2 = 2, \ldots, j_n = n$, and the label of each j is 'Y' if $j \in S$, or 'N' if $j \notin S$.

Proof of (V)

To prove (V), write $\omega(\sigma)$ for the set of outstanding elements of σ. Then we have

$$\begin{aligned} n!\overline{\chi}_n &= \sum_{\sigma} \sum_{j\in\omega(\sigma)} \phi(j) \\ &= \sum_{j=1}^{n} \phi(j) \{ \sum_{\sigma} \sum_{j\in\omega(\sigma)} 1 \}. \end{aligned}$$

The inner sum is the number of permutations that have a given outstanding element j, and by part (III) above that number is $n!/j$. ∎

If we take $\phi(j) = 1$, or j, or j^2, for all j, we find respectively that the mean of the number of outstanding elements is H_n, of their sum is n, and of the sum of their squares is $\binom{n+1}{2}$. If $\phi(j) = 1/j$ we find that the mean value of the sum of the reciprocals of the outstanding elements of n-permutations is $\sum_{j=1}^{n} \frac{1}{j^2} \sim \pi^2/6$.

In fact, we can find not only the average of χ but its complete distribution. Indeed if $f(n, q)$ denotes the number of n-permutations σ for which $\chi(\sigma) = q$, then

$$\sum_{q} f(n, q) t^q = (n-1)! t^{\phi(1)} \prod_{j=2}^{n} \left(1 + \frac{t^{\phi(j)}}{j-1}\right) \qquad (n \geq 1). \tag{2}$$

This follows from the easy recurrence

$$f(n, q) = (n-1) f(n-1, q) + f(n-1, q - \phi(n)) \qquad (n \geq 2;\ f(1, q) = \delta_{q,\phi(1)}).$$

Example. Take $\phi(j) = \log j$. In that case (2) leads to the Dirichlet series generating function

$$\frac{1}{(n-1)!} \sum_{q\geq 1} \frac{h(n, q)}{q^s} = \prod_{j=2}^{n} \left(1 + \frac{1}{j^s (j-1)}\right), \tag{3}$$

in which $h(n, q)$ is now the number of n-permutations in which the *product* of the outstanding elements is equal to q. From (3) we see at once that for each q, $h(n, q)/(n-1)!$ is *constant* for all $n \geq q$, and that if we denote that constant by $h(q)$, then we have the generating function

$$\sum_{q=1}^{\infty} \frac{h(q)}{q^s} = \prod_{j=2}^{\infty} \left(1 + \frac{1}{j^s (j-1)}\right).$$

The numbers $\{h(q)\}_{q=1}^{20}$ are

$$1\,, 1\,, \frac{1}{2}\,, \frac{1}{3}\,, \frac{1}{4}\,, \frac{7}{10}\,, \frac{1}{6}\,, \frac{10}{21}\,, \frac{1}{8}\,, \frac{13}{36}\,, \frac{1}{10}\,, \frac{151}{330}\,, \frac{1}{12}\,, \frac{19}{78}\,, \frac{11}{56}\,, \frac{22}{105}\,, \frac{1}{16}\,, \frac{193}{680}\,, \frac{1}{18}\,, \frac{169}{684}$$

Proof of (VI)

Each monomial μ' in f_{n-1}, coming from a permutation σ', generates a unique monomial $\mu = x_n\mu'$ in f_n, corresponding to the permutation σ whose first $n-1$ values agree with those of σ', and whose nth value is n.

Each monomial μ' in f_{n-1}, coming from a permutation σ', generates, in f_n, the monomial $\mu = (n-1)\mu'$, because the same monomial μ' is contributed to f_n by all of the $n-1$ permutations σ of n letters for which $\sigma(n) \neq n$ and the first $n-1$ values of σ have the same pattern as σ'. Hence $f_n = (x_n + n - 1)f_{n-1}$, and (VI) is proved. This argument, as well as some of the previous ones, might have been phrased in terms of the inversion tables of the permutations.

References

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