Midterm 2 - practice problems

1. Use an appropriate tangent line approximation to estimate $(8.1)^{\frac{1}{3}}$

   Solutions. We will use the tangent line approximation for $y = x^{\frac{1}{3}}$ at $x = 8$.

$$
\begin{aligned}
f'(x) &= \frac{1}{3}x^{-\frac{2}{3}} \\
&= \frac{1}{3x^{\frac{2}{3}}} \\
f'(8) &= \frac{1}{12}
\end{aligned}
$$

   Since $f(8) = 2$ we then have that the linear approximation (or equation of the tangent line) to $y = x^{\frac{1}{3}}$ at $x = 8$ is

$$L(x) = \frac{1}{12}(x-8) + 2$$

   Plugging in 8.1 we get that

$$
\begin{aligned}
L(8.1) &= \frac{1}{12}(.1) + 2 \\
&= \frac{1}{12}\frac{1}{10} + 2 \\
&= 2 + \frac{1}{120} \\
&= \frac{241}{120}
\end{aligned}
$$

2. The length of a side of a cube is measured to be 10 cm with a possible error of $+-.1$ cm. What is the maximum possible error if this measurement is used to determine the volume of the cube?

   Solution: Let x be the length of a side of the cube.

   The volume is $V = x^3$ which gives us that

$$dV = 3x^2 dx$$

   Plugging in $dx = .1$, $x = 10$ we have

$$
\begin{aligned}
dV &= 3(10)^2(.1) \\
&= 30
\end{aligned}
$$

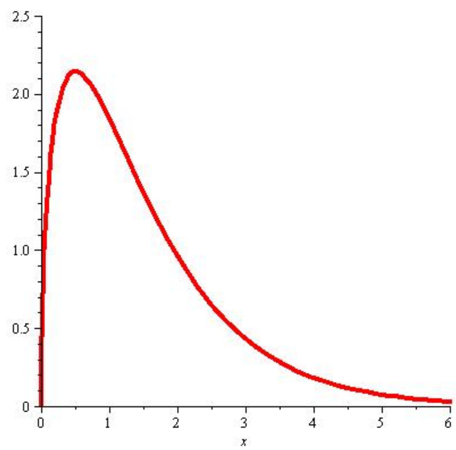
   The maximum error is 30 cm^3.

3. Sketch the graph of a function $y = f(x)$ such that

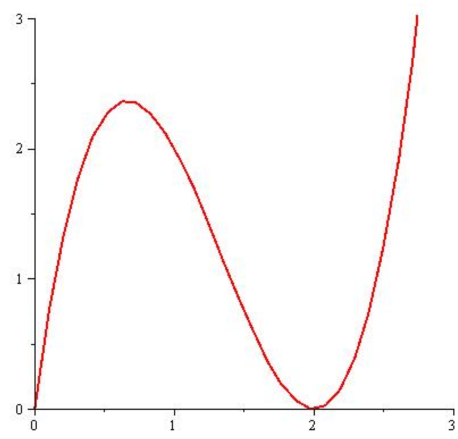
   (a) $f(x) \geq 0$ for all x.
   (b) $f(0) = 0$.
   (c) $\lim_{x\to\infty} f(x) = 0$
   (d) $f'(x) < 0$ when $1 \leq x \leq 3$.
   (e) $f''(x) > 0$ when $2 < x < 4$

   Solutions:

   Here is the graph of one possible solution.



4. The graph of $y = f(x)$ is given below.



(I.)

   (a) When is $f'(x) > 0$?
   (b) When is $f''(x) < 0$?

(c) What are the critical points of f?

(d) What are the inflection points of f?

Solution:

(a) When $0 < x < .75$, or when $2 < x < 3$.

(b) When $0 < x < 1.4$

(c) .75 and 2.

(d) 1.4.

5. Find the absolute maximum and minimum of $x^3 - 6x^2 + 9x + 3$ on the interval $[0, 4]$.

Solution:

$$f(x) = x^3 - 6x^2 + 9x + 3$$

$$\begin{aligned} f'(x) &= 3x^2 - 12x + 9 \\ &= 3(x^2 - 4x + 3) \end{aligned}$$

To find the critical numbers we set $f'(x) = 0$

$$\begin{aligned} 0 &= 3(x^2 - 4x + 3) \\ &= 3(x-3)(x-1) \end{aligned}$$

So the critical numbers are $x = 1$ and $x = 3$. Now plug in the critical numbers and endpoints.

$$\begin{aligned} f(1) &= 7 \\ f(3) &= 3^3 - 6 \cdot 3^2 + 27 + 3 \\ &= 3 \\ f(0) &= 3 \\ f(4) &= 4^3 - 6 \cdot 4^2 + 36 + 3 \\ &= 64 - 48 + 36 + 3 \\ &= 55. \end{aligned}$$

So the absolute maximum is 55 and is obtained at x=4 and the absolute minimum is 3 and is obtained at the points $x = 0$ and $x = 3$.

6. For each of the following functions find the critical numbers and determine whether each is a local max, a local min, or neither.

(a) $3x^4 - 4x^3 + 7$

Solution:

$$\begin{aligned} f'(x) &= 12x^3 - 12x^2 \\ 0 &= 12x^3 - 12x^2 \\ &= 12(x^3 - x^2) \\ &= 12x^2(x-1) \end{aligned}$$

So the critical numbers are 0 and 1.

$$f''(x) = 36x^2 - 24x$$

so

$$f''(1) = 12 > 0$$

therefore f is concave up at $x = 1$ and thus **1 is a local minimum**.

$$f''(0) = 0$$

so the second derivative test fails for $x = 0$.

However, since $f'(x) = 12x^2(x-1)$ we see that $f'(x) < 0$ on the entire interval $(\infty, 1)$. In particular f is decreasing straight through $x = 0$ and **0 is neither a local max or local min.**

(b) $x^{\frac{1}{3}}(x-1)^{\frac{2}{3}}$

Solution:

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-\frac{2}{3}}(x-1)^{\frac{2}{3}} + \frac{2}{3}x^{\frac{1}{3}}(x-1)^{-\frac{1}{3}} \\ &= \frac{1}{3}\left[\frac{(x-1)^{\frac{2}{3}}}{x^{\frac{2}{3}}} + \frac{2x^{\frac{1}{3}}}{(x-1)^{\frac{1}{3}}}\right] \\ &= \frac{x-1+2x}{3x^{\frac{2}{3}}(x-1)^{\frac{1}{3}}} \\ &= \frac{3x-1}{3x^{\frac{2}{3}}(x-1)^{\frac{1}{3}}} \end{aligned}$$

The critical numbers are where either $f'(x) = 0$ or where the derivative is not defined. So the critical numbers are 0, $\frac{1}{3}$, and 1.

Here it will be easier to use the first derivative test. We make the table

	$3x-1$	$x^{\frac{2}{3}}$	$(x-1)^{\frac{1}{3}}$	$f'(x)$
$x < 0$	$(--)$	$(++)$	$(--)$	$(++)$
$0 < x < \frac{1}{3}$	$(--)$	$(++)$	$(--)$	$(++)$
$\frac{1}{3} < x < 1$	$(++)$	$(++)$	$(--)$	$(--)$
$1 < x$	$(++)$	$(++)$	$(++)$	$(++)$

Where the plus or minus signs in the table tell us the sign of each quantity in each interval. We see that $f'(x) > 0$ to the right and left of 0 so **0 is neither a local max or local min**.

At $\frac{1}{3}$ the derivative changes from being positive to negative (i.e. the function increases up to $\frac{1}{3}$ and then starts decreasing, so **1/3 is a local max**.

Similarly **1 is a local min.**

7. Suppose that
$$f(x) = x^2 + ax + b$$
for some constants a and b. If $x = 1$ is a critical number of f and $f(1) = 6$ what are a and b?

Solution:
$$f'(x) = 2x + a$$
Since 1 is a critical point $f'(1) = 0$ so we have
$$0 = f'(1) = 2 + a$$
So $a = -2$.

Since $f(1) = 6$ we then have
$$6 = f(1) = (1)^2 - 2(1) + b$$
so $b = 7$.

8. Is there a differentiable function such that $f(0) = 0$, $f(1) = 7$, and $f'(x) \leq 5$? If so then find one, if not they explain why not.

Solution: There is no such function.

Suppose that we have a function such that $f(0) = 0$ and $f(1) = 7$. Then by the mean value theorem there is a number c between 0 and 1 such that
$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 7$$
So it is impossible for such a function to have $f'(x) \leq 5$.

9. Find the limits

(a) $\lim_{x\to\infty} \frac{2x}{\sqrt{3x^2 - 3}}$

Solution:
$$\lim_{x\to\infty} \frac{2x}{\sqrt{3x^2 - 3}} = \lim_{x\to\infty} \left(\frac{2x}{\sqrt{3x^2 - 3}}\right) \cdot \left(\frac{\frac{1}{x}}{\frac{1}{x}}\right)$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{3x^2-3}\sqrt{\frac{1}{x^2}}}$$
$$= \lim_{x\to\infty} \frac{2}{\sqrt{(3x^2-3)(\frac{1}{x^2})}}$$
$$= \lim_{x\to\infty} \frac{2}{\sqrt{3-\frac{3}{x^2}}}$$
$$= \frac{2}{\sqrt{3}}$$

(b) $\lim_{x\to\infty} \sqrt{4x^2-6} - x$

Solution: This is the indeterminate difference $\infty - \infty$. We multiply and divide by the conjugate.

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{4x^2-6} - x &= \lim_{x\to\infty} \frac{\left(\sqrt{4x^2-6}-x\right)\cdot\left(\sqrt{4x^2-6}+x\right)}{\sqrt{4x^2-6}+x} \\
&= \lim_{x\to\infty} \frac{4x^2-6-x^2}{\sqrt{4x^2-6}+x} \\
&= \lim_{x\to\infty} \frac{3x^2-6}{\sqrt{4x^2-6}+x} \\
&= \lim_{x\to\infty} \frac{\left(3x^2-6\right)}{\left(\sqrt{4x^2-6}+x\right)} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \\
&= \lim_{x\to\infty} \frac{3x-\frac{6}{x}}{\sqrt{4-\frac{6}{x^2}}+1}
\end{aligned}$$

The limit on the bottom is 3 and on the top is ∞, so the total limit is ∞.

10. let $f(x) = \frac{x+1}{x-7}$

(a) What is the domain of f?

Solution: All x except 7.

(b) What are the vertical and horizontal asymptotes of f?

Solution: There is a vertical asymptote at $x = 7$

When $x > 7$, $x - 7 > 0$ and $x + 1 > 0$ so

$$\lim_{x\to 7^+} \frac{x+1}{x-7} = \infty$$

When $1 < x < 7, x - 7 < 0$ and $x + 1 > 0$ so

$$\lim_{x\to 7^-} \frac{x+1}{x-7} = -\infty$$

To find the horizontal asymptotes we see that

$$\lim_{x\to\infty} \frac{x+1}{x-7} = 1$$

and that

$$\lim_{x\to-\infty} \frac{x+1}{x-7} = 1$$

So the function has horizontal asymptotes at 1 as it goes to $+\infty$ and $-\infty$.

(c) When is f increasing? decreasing?

Solution:

$$\begin{aligned} f'(x) &= \frac{(x-7)-(x+1)}{(x-7)^2} \\ &= \frac{-8}{(x-7)^2} \end{aligned}$$

So the derivative is always negative, and the function is always decreasing.

(d) When is f concave up? concave down?

Solution:

We rewrite

$$f'(x) = -8(x-7)^{-2}$$

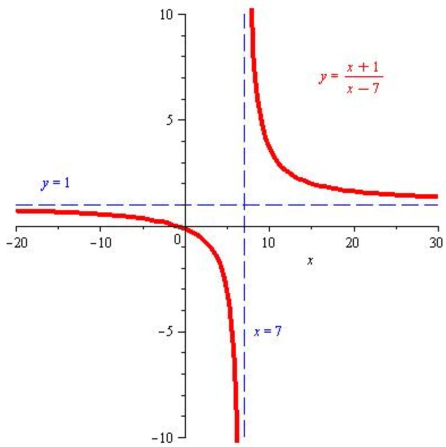
so

$$\begin{aligned} f''(x) &= 16(x-7)^{-3} \\ &= \frac{16}{(x-7)^3} \end{aligned}$$

So we have that $f''(x) > 0$ when $x > 7$ and $f''(x) < 0$ when $x < 7$. So f is concave up for $x > 7$ and concave down when $x < 7$.

(e) Use the information from (a)-(d) to sketch the graph of f.

Solution: Here is the graph of the function



11. A farmer wants to enclose a rectangular area of 64 square feet along the shore of a river and then divide it into two pens with a fence down the middle perpendicular to the river. (There is no fence along the edge of the river). The fencing for the perimeter fence costs \$5 per foot. The fencing used to divide the pen costs \$10 per foot. What dimensions should the farmer make the pen to minimize the cost of the fence?

    Solution:

    Let x be the length of the side of the fence parallel to the river and y be the length of the side perpendicular to the river.

    Then we want to minimize the Cost, whose formula in terms of x and y is

    $$Cost = 5x + 20y$$

    We also have the restriction that the area is $64ft^2$.

    $$64 = xy$$

    Plugging this into the equation for cost we obtain

    $$\begin{aligned} C(x) &= 5x + 20\left(\frac{64}{x}\right) \\ &= 5x + 1280x^{-1} \end{aligned}$$

    To find the minimum of C we find the critical points.

    $$C'(x) = 5 - 1280x^{-2}$$

$$\begin{aligned} 0 &= 5 - \frac{1280}{x^2} \\ \frac{1280}{x^2} &= 5 \\ 1280 &= 5x^2 \\ 256 &= x^2 \end{aligned}$$

So we have $x = 16$ or -16. Since x is a length it must be positive so $x = 16$.

We can also see that $C'(x) < 0$ when $x < 16$ and $C'(x) > 0$ when $x > 16$ so that, by the first derivative test, an absolute minimum of C occurs at $x = 16$.

So the dimensions which minimize the cost are $x = 16, y = 4$.

12. Find the most general anti-derivative of the following functions with respect to x

    (a) $4\sin(x) + 7$
    Solution:
    $-4\cos(x) + 7x + c$

    (b) $x^5 - x^3 + 7x^2$.
    Solution:
    $\frac{x^6}{6} - \frac{x^4}{4} + \frac{7x^3}{3} + c$

    (c) $\sec^2(x)$
    Solution:
    $\tan(x) + c$.

13. The acceleration of a particle moving in a straight line with $t \geq 0$ is given by

$$a(t) = 2t - 4$$

    (a) If the particle was at rest when $t = 0$, find $v(t)$.
    Solution:
    $v(t)$ is the anti-derivative of $a(t)$ so

$$v(t) = t^2 - 4t + c$$

    Since the particle was at rest when $t = 0$ we have

$$0 = v(0) = 0^2 - 4(0) + c = c$$

    so

$$v(t) = t^2 - 4t$$

(b) When was the particle moving to the left? To the right?

Solution: The particle is moving to the left when $v(t) < 0$ and to the right when $v(t) > 0$.

$$\begin{aligned} v(t) &= t^2 - 4t \\ &= t(t-4) \end{aligned}$$

So $v(t) < 0$ when $0 < t < 4$ and

$v(t) > 0$ when $t > 4$.

So the particle is moving to the left when $0 < t < 4$ and is moving to the right when $t > 4$.

(c) When was the particle speeding up/slowing down?

Solution: The particle is speeding up when the $v(t)$ and $a(t)$ have the same sign and is slowing down when they have opposite signs.

Since $a(t) = 2t - 4$, the acceleration is negative when $0 < t < 2$ and is then is positive when $t > 2$.

Combining this with the part (b) we have that the particle is speeding up when $0 < t < 2$ and when $t > 4$ and is slowing down when $2 < t < 4$.

(d) What is the total distance travelled by the particle from $t = 0$ to $t = 5$?

Solution: We need the displacement which is the anti-deriviative of $v(t)$.

$$\begin{aligned} s(t) &= \frac{t^3}{3} - \frac{4t^2}{2} + c \\ &= \frac{t^3}{3} - 2t^2 + c \end{aligned}$$

From part (b) we know that the particle moves left from $t = 0$ to $t = 4$ so the distance travelled in that time interval is

$$\begin{aligned} s(0) - s(4) &= c - \frac{4^3}{3} + 24^2 - c \\ &= 32 - \frac{64}{3} \\ &= \frac{32}{3} \end{aligned}$$

From $t = 4$ to $t = 5$ the particle is moving to the right so the distance travelled in that time interval is

$$s(5) - s(4) = \frac{5^3}{3} - 25^2 + c - \frac{64}{3} + 32 - c$$

$$\begin{aligned}
&= \frac{125}{3} - 50 - \frac{64}{3} + 32 \\
&= \frac{61}{3} - 18 \\
&= \frac{61}{3} - \frac{54}{3} \\
&= \frac{7}{3}
\end{aligned}$$

So the total distance travelled is

$$\frac{32}{3} + \frac{7}{3} = \frac{39}{3} = 13$$

14. Estimate $\int_1^5 \frac{2}{1+x^2} dx$. Using $n = 4$ rectangles and left endpoints.

$$\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = 1$$

So the endpoints of our intervals are

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= 2 \\
x_2 &= 3 \\
x_3 &= 4 \\
x_4 &= 5
\end{aligned}$$

And

$$\begin{aligned}
L_4 &= \Delta x \left(f(x_0) + f(x_1) + f(x_2) + f(x_3) \right) \\
&= 1(1 + \frac{2}{5} + \frac{2}{10} + \frac{2}{17}) \\
&= 1 + \frac{68}{170} + \frac{34}{170} + \frac{20}{170} \\
&= \frac{292}{170}
\end{aligned}$$