Math 425 Final exam practice problems

1. Find the solution to

$$u_{tt} = u_{xx} + 2u_x + u.$$
$$u(x,0) = x^2$$
$$u_t(x,0) = 0$$

Solution:

Set $v = ue^x$, then by the product rule,

$$v_x = u_x e^x + ue^x$$
$$v_{xx} = u_{xx}e^x + 2u_x e^x + ue^x$$
$$v_{tt} = u_{tt}e^x$$

So,

$$v_{xx} = (u_{xx} + 2u_x + u)e^x = u_{tt}e^x = v_{tt}$$

So v is a solution to the wave equation with wave speed 1. The general solution is

$$v = f(x-t) + g(x+t)$$

The initial conditions are

$$v(x,0) = u(x,0)e^x = x^2e^x$$
$$v_t(x,0) = u_t(x,0)e^x = 0$$

Which is

$$f(x) + g(x) = x^2e^x$$
$$-f'(x) + g'(x) = 0$$

The first equation becomes

$$f'(x) + g'(x) = (x^2e^x)'$$

Adding the equations we have

$$2g'(x) = (x^2e^x)'$$

So

$$g(x) = \frac{1}{2}x^2e^x + c_1$$

Similarly

$$f(x) = \frac{1}{2}x^2e^x + c_2$$

From here we see that $c_1 + c_2 = 0$, So our solution is

$$v(x,t) = \frac{1}{2}[(x+t)^2 e^{x+t} + (x-t)^2 e^{x-t}] = \frac{e^x}{2}\left[(x+t)^2 e^t + (x-t)^2 e^{-t}\right]$$

So

$$u(x,t) = e^{-x} v = \frac{1}{2}\left[(x+t)^2 e^t + (x-t)^2 e^{-t}\right]$$

2. Is there a solution to the boundary value problem

$$u''(x) + 3u'(x) = x$$

$$u'(0) + 3u(0) = u'(1) + 3u(1)?$$

If so then find one, if not then prove that no such solution exists.

Solution: There is no solution because if there were then we could integrate both sides of the equation

$$\int_0^1 (u''(x) + 3u'(x))dx = \int_0^1 x dx$$

$$u'(x) + 3u(x)|_0^1 = \frac{1}{2}$$

$$(u'(1) + 3u(1)) - (u'(0) + 3u(0)) = \frac{1}{2}$$

But this is impossible since the left hand side of the equation is zero by the boundary conditions.

3. Let

$$f(x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x \le \pi \end{cases}$$

(a) Find the sine series of f on $[0, \pi]$.

Solution:

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx \\ &= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\sin(nx)dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin(nx)dx\right] \\ &= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx)|_0^{\pi/2} + \int_0^{\frac{\pi}{2}} \frac{\cos(nx)}{n}dx - \left(\frac{\pi - x}{n}\right)\cos(nx)|_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(nx)}{n}\right] \\ &= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{\sin(nx)}{n^2}|_0^{\pi/2} + \frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) - \frac{\sin(nx)}{n^2}|_{\pi/2}^{\pi}\right] \\ &= \frac{4\sin\left(\frac{n\pi}{2}\right)}{\pi n^2} \end{aligned}$$

When n is even, $\sin\left(\frac{n\pi}{2}\right)$ is zero. So we only need to consider when n is odd, that is when $n = 2m - 1$. But then

$$\sin\left(\frac{(2m-1)\pi}{2}\right) = \sin\left(\left(m - \frac{1}{2}\right)\pi\right)$$

Which is 1 when m is odd and -1 when m is even, so

$$A_n = \frac{4}{\pi}\frac{(-1)^{n+1}}{(2n-1)^2}$$

and the Fourier series is

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin((2n-1)x)$$

(b) What is the solution to the boundary value problem

$$u_{tt} = u_{xx}$$

$$u(0,t) = u(\pi,t) = 0$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = 0?$$

Solution: This is the wave equation with Dirichlet boundary conditions. The solution is

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos((2n-1)x)\sin((2n-1)x)$$

4. The full Fourier series of the function

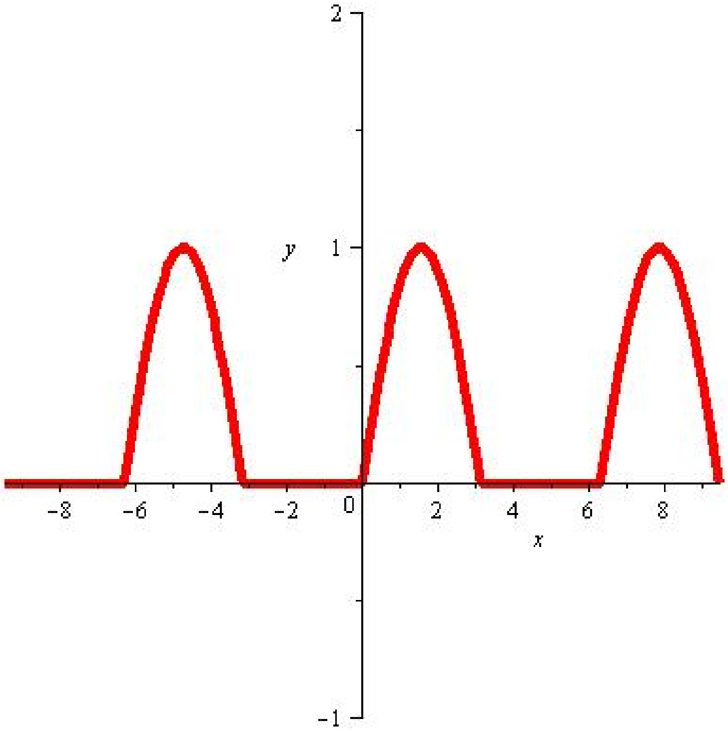
$$f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ \sin(x) & 0 \leq x \leq \pi \end{cases}$$

is

$$\frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}.$$

(a) Draw the graph of the function which the Fourier series converges to pointwise on $-3\pi \leq x \leq 3\pi$.

Solution:



(b) Find the solution to the Laplace's equation on the unit disc with boundary value f

$$\Delta u = 0$$

$$u = f(\theta)$$

Solution: In polar coordinates the solution is

$$\frac{1}{\pi} + \frac{r}{2}\sin(\theta) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{r^{2n}\cos(2n\theta)}{4n^2-1}$$

(c) Use the Fourier series to find

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

Solution: We want to use the convergence theorem for Fourier series to find this sum, so we need to plug in an appropriate value for x to get the desired sum. We see this will happen if

$$\cos(2nx) = 1$$

for every n, which is the case if $x = 0$. Then by the pointwise convergence theorem for Fourier series we have that

$$f(0) = \frac{1}{\pi} + \frac{1}{2}\sin(0) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$$

$$0 = \frac{1}{\pi} + 0 - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$$

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$$

(d) Use the Fourier series to find

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

Now we want

$$\cos(2nx) = (-1)^n$$

We see this is satisfied when $x = \frac{\pi}{2}$, since we then have

$$\cos(n\pi) = (-1)^n$$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$$

so by the pointwise convergence theorem,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{1}{2} - \frac{\pi}{4}$$

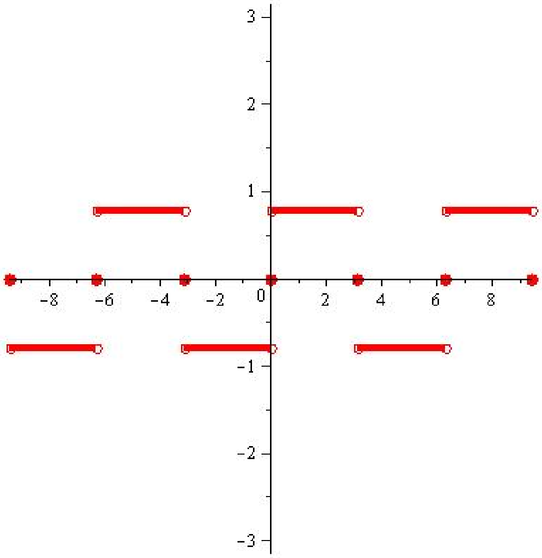
5. The full Fourier series of the function

$$f(x) = \begin{cases} -\frac{\pi}{4} & -\pi \le x < 0 \\ \frac{\pi}{4} & 0 \le x \le \pi \end{cases}$$

is

$$\sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{(2n-1)}$$

(a) Draw the graph of the Fourier series on $-3\pi \le x \le 3\pi$.



(b) Compute an infinite sum by applying Parseval's equation to the Fourier series.

Solution:

Parseval's equation gives that

$$\pi \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \int_{-\pi}^{\pi} |f(x)|^2 dx$$

But

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \int_{-\pi}^{\pi} \frac{\pi^2}{16} dx = \frac{\pi^3}{8}$$

So we obtain

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

(c) Find the function of the form

$$p(x) = a_1 \cos(x) + b_1 \sin(x) + b_2 \sin(3x)$$

which minimizes the quantity $\int_{-\pi}^{\pi} \left(f(x) - p(x)\right)^2$.
Solution: This is just the Fourier coefficients so

$$a_1 = 0$$

$$b_1 = 1$$

$$b_2 = \frac{1}{3}$$

(d) Find the Fourier sine series of the function $g(x) = \frac{\pi}{4}$ on $0 \leq x \leq \pi$. Justify your answer.

Solution: f is just the odd extension of g. Therefore, the sine series for g is just the fourier series for f

$$\sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{(2n-1)}$$

6. (a) Let f and g be nonnegative functions on some interval $[a, b]$. Show that

$$\|f - g\|^2 \leq \|f\|^2 + \|g\|^2.$$

Solution:

$$\begin{aligned} \|f-g\|^2 &= \langle f-g, f-g\rangle = \langle f, f\rangle - 2\langle f, g\rangle + \langle g, g\rangle \\ &= \|f\|^2 - 2\langle f, g\rangle + \|g\|^2 \end{aligned}$$

But

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx \geq 0$$

Since f and g are both nonnegative. Plugging this into the above gives

$$\|f - g\|^2 \leq \|f\|^2 + \|g\|^2$$

(b) Show that for any (not nec. nonnegative) functions $\|f - g\|^2 = \|f\|^2 + \|g\|^2$ if and only if f and g are orthogonal.
Solution:
From the above we have

$$\|f - g\|^2 = \|f\|^2 - 2\langle f, g\rangle + \|g\|^2$$

so

$$\|f - g\|^2 = \|f\|^2 + \|g\|^2$$

if and only if

$$\langle f, g\rangle = 0$$

which is the definition of f and g being orthogonal.

(c) Use part (b) to show that

$$\int_{-\pi}^{\pi} (\sin(mx) - \cos(nx))^2 \, dx = 2\pi$$

for any integers m and n.

Solution: We know that $\sin(nx)$ and $\cos(mx)$ are orthogonal on $[-\pi, \pi]$, so from part b we know that

$$\| \sin(nx) - \cos(mx)\|^2 = \| \sin(nx)\|^2 + \| \cos(mx)\|^2$$

Which by definition says

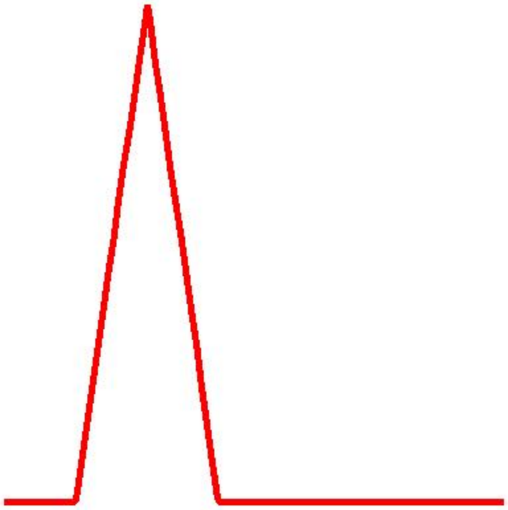
$$\int_{-\pi}^{\pi} (\sin(mx) - \cos(nx))^2 \, dx = \int_{-\pi}^{\pi} \sin^2(nx)dx + \int_{-\pi}^{\pi} \cos^2(mx)dx = \pi + \pi = 2\pi$$

7. For each positive integer n define

$$f_n(x) = \begin{cases} 0 & 0 \le x < \frac{1}{n} \\ n^3x - n^2 & \frac{1}{n} \le x < \frac{2}{n} \\ -n^3x + 3n^2 & \frac{2}{n} \le x < \frac{3}{n} \\ 0 & \frac{3}{n} \le x \le 1 \end{cases}$$

(a) Sketch the graph of f_n.

Solution: The graph of the function is zero except for a triangular spike starting at the point $(\frac{1}{n}, 0)$, reaching its top at the point $(\frac{2}{n}, n^2)$ and then returns the x-axis at $(\frac{3}{n}, 0)$. An example is drawn below. As n gets larger the spike moves to the left towards zero and becomes thinner and taller.



(b) Show that the sequence of functions f_n converges pointwise to the zero function on the interval $[0,1]$.

Solution: Fix a point in $[0,1]$ call it x. By the definition of pointwise convergence we just need to show that the sequence of numbers $\{f_n(x)\}$ converges to zero. In fact, if we let n be large enough so that $x > \frac{3}{n}$, then x will be to the right of the spike, so $f_n(x) = 0$ for all $n > \frac{3}{x}$ and therefore

$$\lim_{n\to\infty} f_n(x) = 0.$$

(c) Show the the sequence f_n does not converge in the mean.

Solution:

If a function converges pointwise and converges in the mean then it must converge in the mean to its pointwise limit. Therefore, we just need to show that f_n does not converge in the mean to zero. In other words we need to show that

$$\lim_{n\to\infty} \int_0^1 |f_n(x) - 0|^2 dx \neq 0$$

We compute

$$\begin{aligned}
\int_0^1 |f_n(x) - 0|^2 dx &= \int_{\frac{1}{n}}^{\frac{2}{n}} (n^3x - n^2)^2 dx + \int_{\frac{2}{n}}^{\frac{3}{n}} (-n^3x + 3n^2)^2 dx \\
&= \int_{\frac{1}{n}}^{\frac{2}{n}} n^6x^2 - n^5x + n^4 dx + \int_{\frac{2}{n}}^{\frac{3}{n}} n^6x^2 - 6n^5x + 9n^4 dx \\
&= \frac{n^6}{3}\left(\frac{8}{n^3} - \frac{1}{n^3}\right) - \frac{n^5}{2}\left(\frac{4}{n^2} - \frac{1}{n^2}\right) + n^3 \\
&\qquad + \frac{n^6}{3}\left(\frac{27}{n^3} - \frac{8}{n^3}\right) - \frac{6n^5}{2}\left(\frac{9}{n^2} - \frac{4}{n^2}\right) + 9n^3 \\
&= \frac{7}{3}n^3 - \frac{3}{2}n^3 + n^3 + \frac{19}{3}n^3 - 15n^3 + 9n^3 \\
&= \left(\frac{43}{6} - 5\right) n^3 \\
&= \frac{13}{6}n^3
\end{aligned}$$

The limit as $n \to \infty$ is ∞, not zero, so the sequence does not converge in the mean.

8. Find the solution to

$$u_t = u_{xx}$$

$$u(0,t) = 0 \qquad u(\pi,t) = \frac{\pi}{2}$$

$$u(x,0) = 0$$

Solution: We use the method of "shifting the data". Let $U(x,t) = \frac{x}{2}$, then U is a solution to

$$U_t = U_{xx}$$

$$U(0,t) = 0 \qquad U(\pi,t) = \frac{\pi}{2}$$

So if we let $w = u - U$, the w solves

$$w_t = w_{xx}$$

$$w(0,t) = 0 = w(\pi,t)$$

$$w(x,0) = -\frac{x}{2}$$

To solve this equation we need the Fourier sine series for $-x/2$ on $[0,\pi]$, which is just

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)$$

(Just the Fourier series for x multiplied by $-1/2$.) So we have

$$w(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2 t} \sin(nx)$$

Then $u = w + U$ so

$$u(x,t) = \frac{x}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2 t} \sin(nx)$$

9. Find the solution to Laplace's equation on the square

$$0 < x < \pi \qquad 0 < y < \pi$$

Satisfying the boundary conditon

$$u_x(0,y) = 0 \qquad u_x(\pi,y) = 0 \qquad u(x,\pi) = 0 \qquad u(x,0) = \cos(2x)$$

Solution: We seperate variables

$$u(x,y) = F(x)G(y)$$

then Laplace's equation says that

$$F''(x)G(y) + F(x)G''(y) = 0$$

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = \lambda$$

The boundary conditions in x are

$$F'(0) = 0 \qquad F'(\pi) = 0$$

The boundary conditions in y are

$$G(1) = 0$$

$$\cos(2x) = u(x,0) = F(x)G(0)$$

So we notice that in this particular case we can take $F(x) = \cos(2x)$ and

$$G(0) = 1 \qquad G(\pi) = 0$$

and satisfy all the boundary conditions. Then we have that $\lambda = -4$, so our equation for G is

$$G''(y) = 4G(y)$$

So

$$G(y) = c_1 e^{2y} + c_2 e^{-2y}$$

Applying the boundary conditions for G

$$\begin{gathered} c_1 + c_2 = 1 \\ c_1 e^{2\pi} + c_2 e^{-2\pi} = 0 \end{gathered}$$

so

$$\begin{gathered} c_1 e^{2\pi} + (1 - c_1)e^{-2\pi} = 0 \\ c_1(e^{2\pi} - e^{-2\pi}) = -e^{-2\pi} \\ c_1 = -\frac{e^{-2\pi}}{e^{2\pi} - e^{-2\pi}} \end{gathered}$$

and

$$c_2 = \frac{e^{2\pi}}{e^{2\pi} - e^{-2\pi}}$$

Therefore

$$\begin{aligned} u(x,y) &= F(x)G(y) \\ &= \cos(2x)\left(\left(-\frac{e^{-2\pi}}{e^{2\pi} - e^{-2\pi}}\right)e^{2y} + \left(\frac{e^{2\pi}}{e^{2\pi} - e^{-2\pi}}\right)e^{-2y}\right) \\ &= \frac{\cos(2x)}{e^{2\pi} - e^{-2\pi}}\left(-e^{2(y-\pi)} + e^{2(\pi-y)}\right) \end{aligned}$$

10. Find all the nonnegative harmonic functions on the unit disc such that

$$u(0,0) = 0$$

Solution:

The only such function is the constant function $u(x, y) = 0$. To see this let u be a nonnegative function on the unit disk such that $u(0, 0) = 0$. Then 0 is a minimum for u, so u is a harmonic function on the disc which achieves its minimum in the interior of the disc. By the strong maximum principle (or the mean value property of harmonic functions) u must be constant. Since it is already 0 at the origin, it must be zero everywhere.

11. Let D be a bounded region in 3-space. A constant λ is called a Dirichlet eigenvalue of D if there is a function other than the zero function u on D such that

$$\Delta u = \lambda u$$

that satisfies the Dirchlet boundary condition

$$u = 0$$

on ∂D. The function u is called an eigenvector for D

(a) Show that $\lambda < 0$.

Solution:

Let $\Delta u = \lambda u$.

First we show that $\lambda \leq 0$. Apply Green's first identity to the pair (u,u).

$$\iint_{\partial D} u\frac{\partial u}{\partial n} = \iiint_D |\nabla u|^2 + \iint_D u\Delta u$$

Since u satisfies Dirichlet boundary conditions we get

$$0 = \iiint_D |\nabla u|^2 + \iiint_D \lambda u^2$$

so

$$\lambda = -\frac{\iiint_D |\nabla u|^2}{\iiint_D u^2}$$

So $\lambda \leq 0$ since both terms on the right are nonnegative (note that the denominator is not zero, because u is assumed to not be the zero function.)

Now to see that $\lambda < 0$ suppose that $\lambda = 0$, then

$$\iiint_D |\nabla u|^2 = 0$$

Which implies u is constant, but since u is zero on the boundary this implies that u is the zero function which is impossible, so $\lambda < 0$

(b) Show that any two eigenfunctions on D corresponding to different eigenvalues are orthogonal over D.

Solution: Two functions u, v are orthogonal on D if

$$\iiint_D uv = 0$$

Let u_1 and u_2 be Dirichlet eigenfunctions with different corresponding eigenvalues λ_1 and λ_2. In other words

$$\Delta u_1 = \lambda_1 u_1$$

$$\Delta u_2 = \lambda_2 u_2$$

Apply Green's second identity to the pair (u_1, u_2)

$$\iint_{\partial D} \left(u_1 \frac{\partial u_1}{\partial n} - u_2 \frac{\partial u_2}{\partial n} \right) = \iiint_D u_1 \Delta u_2 - u_2 \Delta u_1$$

Applying the equation and boundary conditions gives

$$0 = (\lambda_2 - \lambda_1) \iiint_D u_1 u_2$$

Since $\lambda_1 \neq \lambda_2$, we must have u_1 and u_2 orthogonal.