Math 425 Final exam practice problems

1. Find the solution to

$$u_{tt} = u_{xx} + 2u_x + u.$$
$$u(x,0) = x^2$$
$$u_t(x,0) = 0$$

2. Is there a solution to the boundary value problem

$$u''(x) + 3u'(x) = x$$
$$u'(0) + 3u(0) = u'(1) + 3u(1)?$$

If so then find one, if not then prove that no such solution exists.

3. Let

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x \leq \pi \end{cases}$$

(a) Find the sine series of f on $[0, \pi]$.

(b) What is the solution to the boundary value problem

$$u_{tt} = u_{xx}$$
$$u(0,t) = u(\pi,t) = 0$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = 0?$$

4. The full Fourier series of the function

$$f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ \sin(x) & 0 \leq x \leq \pi \end{cases}$$

is

$$\frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2 - 1}.$$

(a) Draw the graph of the function which the Fourier series converges to pointwise on $-3\pi \leq x \leq 3\pi$.

(b) Find the solution to the Laplace's equation on the unit disc with boundary value f

$$\Delta u = 0$$
$$u = f(\theta)$$

(c) Use the Fourier series to find

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1}.$$

(d) Use the Fourier series to find

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}.$$

5. The full Fourier series of the function

$$f(x) = \begin{cases} -\frac{\pi}{4} & -\pi \le x < 0 \\ \frac{\pi}{4} & 0 \le x \le \pi \end{cases} \quad is \sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{(2n-1)}$$

(a) Draw the graph of the Fourier series on $-3\pi \le x \le 3\pi$.

(b) Compute an infinite sum by applying Parseval's equation to the Fourier series.

(c) Find the function of the form

$$p(x) = a_1 \cos(x) + b_1 \sin(x) + b_2 \sin(3x)$$

which minimizes the quantity $\int_{-\pi}^{\pi} (f(x) - p(x))^2$.

(d) Find the Fourier sine series of the function $g(x) = \frac{\pi}{4}$ on $0 \le x \le \pi$. Justify your answer.

6. (a) Let f and g be nonnegative functions. Show that

$$\|f-g\|^2 \le \|f\|^2 + \|g\|^2.$$

(b) Show that for any (not nec. nonnegative) functions $\|f-g\|^2 = \|f\|^2 + \|g\|^2$ if and only if f and g are orthogonal.

(c) Use part (b) to show that

$$\int_{-\pi}^{\pi} (\sin(mx) - \cos(nx))^2 \, dx = 2\pi$$

for any integers m and n.

7. For each positive integer n define

$$f_n(x) = \begin{cases} 0 & 0 \le x < \frac{1}{n} \\ n^3x - n^2 & \frac{1}{n} \le x < \frac{2}{n} \\ -n^3x + 3n^2 & \frac{2}{n} \le x < \frac{3}{n} \\ 0 & \frac{3}{n} \le x \le 1 \end{cases}$$

(a) Sketch the graph of f_n.

(b) Show that the sequence of functions f_n converges pointwise to the zero function on the interval $[0,1]$.

(c) Show the the sequence f_n does not converge in the mean.

8. Find the solution to

$$u_t = u_{xx}$$
$$u(0,t) = 0 \qquad u(\pi,t) = \frac{\pi}{2}$$
$$u(x,0) = 0$$

9. Find the solution to Laplace's equation on the square

$$0 < x < \pi \qquad 0 < y < \pi$$

Satisfying the boundary conditon

$$u_x(0,y) = 0 \qquad u_x(\pi,y) = 0 \qquad u(x,\pi) = 0 \qquad u(x,0) = \cos(2x)$$

10. Find all the nonnegative harmonic functions on the unit disc such that

$$u(0,0) = 0$$

11. Let D be a bounded region in 3-space. A constant λ is called a Dirichlet eigenvalue of D if there is a non-zero function u on D such that

$$\Delta u = \lambda u$$

that satisfies the Dirchlet boundary condition

$$u = 0$$

on ∂D. The function u is called an eigenvector for D

(a) Show that $\lambda < 0$.

(b) Show that any two eigenfunctions on D corresponding to different eigenvalues are orthogonal over D.