Math 425 Midterm practice problem solutions

1. Find the general solution to $3u_x + u_{xy} = 1$.
2. Let $u(x,t)$ be a solution to
$$u_t = ku_{xx} - ru \qquad 0 < x < L, 0 < t < \infty$$
where k and r are positive constants, with Dirichlet boundary conditons
$$u(0,t) = 0 \qquad u(L,t) = 0$$
   (a) Show that the total energy
$$E(t) = \int_0^L u^2(x,t)dx,$$
   is a decreasing function of t
   (b) Prove that the solution to the above problem (if it exists) is unique.
3. Let $u(x,t)$ $0 \leq x \leq 1$, $0 \leq t \leq \infty$ be a solution to
$$u_t - u_{xx} = 3t^2$$
such that $u(x,0) \leq 1$ and $u(0,t), u(1,t) \leq t^3 + 1$.
Show that $u(x,t) \leq t^3 + 1$ for all $0 \leq x \leq 1$ and $0 \leq t \leq \infty$.
4. The solution $u(x,t)$ to the heat equation
$$u_t = u_{xx} \qquad -\infty < x < \infty, 0 \leq t < \infty$$
with initial condition
$$u(x,0) = x^4$$
is
$$u(x,t) = x^4 + 12tx^2 + 12t^2$$
Use this fact to...
   (a) Find $\int_{-\infty}^{\infty} p^4 e^{-p^2} dp$.
   (b) Find the solution to the heat equation with initial condition $u(x,0) = x^5$.
5. Solve the wave equation with friction
$$u_{xx} = u_{tt} + 2u_t \qquad 0 < x < \pi, t > 0$$
with initial conditions
$$u(x,0) = \sin(x) \qquad u_t(x,0) = 0.$$
(Hint: Look for separated solutions.)

6. Find the eigenvalues and eigenfunctions for

$$u'' + \lambda u = 0$$

with boundary conditions

$$u(0) = 0 \qquad u(1) + u'(1) = 0$$

7. Let $f(x) = x(1-x)$ $0 \leq x \leq 1$.

   (a) Find the fourier sine series of f on $[0, 1]$.

   (b) What is the solution to the wave equation

   $$u_{tt} = u_{xx}$$

   with Dirichlet boundary conditions

   $$u(0, t) = u(1, t)$$

   and initial conditions

   $$u(x, 0) = f(x)$$
   $$u_t(x, 0) = 0$$