Math 425 Midterm practice problems

1. Find the general solution to $3u_x + u_{xy} = 1$.

   Let $v = u_x$, then the equation becomes

$$3v + v_y = 1$$

multiply both sides by the integrating factor e^{3y}

$$e^{3y}(3v + v_y) = e^{3y}$$

$$\frac{d}{dy}(e^{3y}v) = e^{3y}$$

Integrate (with respect to y).

$$e^{3y}v = \frac{1}{3}e^{3y} + f(x)$$

$$u_x = v = \frac{1}{3} + f(x)e^{-3y}$$

Integrating (with respect to x) we obtain

$$u = \frac{x}{3} + F(x)e^{-3y} + G(y)$$

where F and G are arbitrary functions of one variable.

CHECK, that this is a solution.

$$u_x = \frac{1}{3} + F'(x)e^{-3y}$$

$$u_{xy} = -3F'(x)e^{-3y}$$

$$\begin{aligned} 3u_x + u_{xy} &= 1 + 3F'(x)e^{-3y} - 3F'(x)e^{-3y} \\ &= 1 \end{aligned}$$

2. Let $u(x,t)$ be a solution to

$$u_t = ku_{xx} - ru \qquad 0 < x < L, 0 < t < \infty$$

where k and r are positive constants, with Dirichlet boundary conditons

$$u(0,t) = 0 \qquad u(L,t) = 0$$

and initial conditions

$$u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x)$$

(a) Show that the total energy

$$E(t) = \int_0^L u^2(x,t)dx,$$

is a decreasing function of t

$$\begin{aligned}
\frac{dE}{dt} &= \int_0^L \frac{d}{dt}(u^2(x,t))dx \\
&= \int_0^L 2uu_t dx \\
&= \int_0^L 2u(ku_{xx} - ru)dx \\
&= 2k\int_0^L uu_{xx} - r\int_0^L u^2 dx \\
&= 2k\left[uu_x|_{x=0}^{x=L} - \int_0^L (u_x)^2 dx\right] - r\int_0^L u^2 dx \\
&= 2k[0 - \int_0^L (u_x)^2 dx - r\int_0^L u^2 dx \\
&= -2k\int_0^L (u_x)^2 dx - r\int_0^L u^2 dx \\
&\leq 0
\end{aligned}$$

Where the 3rd line is obtained from the second line by plugging in the equation, the 5th line is obtained from the fourth using integration by parts, and the first term in the fifth line vanishes because of the boundary conditions.

(b) Prove that the solution to the above problem (if it exists) is unique.

Let u_1 and u_2 be solutions and set $w = u_2 - u_1$ then

$$\begin{aligned} w_{tt} &= (u_1)_{tt} - (u_2)_{tt} \\ &= k(u_1)_{xx} - r(u_1) - k(u_2)_{xx} + r(u_1) \\ &= kw_{xx} - rw \end{aligned}$$

$$w(0,t) = u_1(0,t) - u_2(0,t) = 0$$

$$w(L,t) = u_1(L,t) - u_2(L,t) = 0$$

So w is also a solution to the problem with intiial conditions

$$w(x,0) = u_1(x,0) - u_2(x,0) = \phi(x) - \phi(x) = 0$$

$$w_t(x,0) = \psi(x) - \psi(x) = 0$$

But then the initial Energy of w is zero. E(0) = 0. Since the energy is decreasing this implies $E(t) \leq 0$ for all t. But then

$$0 \leq \int_0^L w^2(x,t)dx \leq 0$$

This is only possible if $w(x,t) = 0$ for all x and t, which implies $u_1 = u_2$ for all x and t, and the solution is unique.

3. Let $u(x,t)$ $0 \leq x \leq 1$, $0 \leq t \leq \infty$ be a solution to

$$u_t - u_{xx} = 3t^2$$

such that $u(x,0) \leq 1$ and $u(0,t), u(1,t) \leq t^3 + 1$.

Show that $u(x,t) \leq t^3 + 1$ for all $0 \leq x \leq 1$ and $0 \leq t \leq \infty$.

Let $v(x,t) = u - t^3$. Then

$$v_t - v_{xx} = u_t - u_{xx} - 3t^2 = 0$$

So v is a solution to the diffusion equation on the rectangle $0 \leq x \leq 1$, $0 \leq t \leq \infty$. Therefore by the maximum principle its maximum must occur on the boundary of the rectangle.

On the bottom of the rectangle,

$$v(x,0) = u(x,0) \leq 1$$

And on the sides,

$$v(0,t) = u(0,t) - t^3 \leq 1$$

$$v(L,t) = u(L,t) - t^3 \leq 1$$

So $v(x,t) \leq 1$ for all x and t in the rectangle, which implies $u(x,t) \leq t^3 + 1$.

4. The solution $u(x,t)$ to the heat equation

$$u_t = u_{xx} \qquad -\infty < x < \infty, 0 \le t < \infty$$

with initial condition

$$u(x,0) = x^4$$

is

$$u(x,t) = x^4 + 12tx^2 + 12t^2$$

Use this fact to...

(a) Find $\int_{-\infty}^{\infty} p^4 e^{-p^2} dp$.
We have the formula for the solution to the diffusion equation in terms of the source function

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} y^4 dy$$

So by uniqueness of solutions we get that the two functions are equal for every x and t.

$$x^4 + 12tx^2 + 12t^2 = u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} y^4 dy$$

Setting $x = 0$ and $t = 1$ we obtain

$$12 = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-(y)^2/4} y^4 dy$$

Setting $p = y/2$, we have $p^4 = y^4/16$ and $2dp = dy$. So we obtain

$$12 = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2} 32p^4 dp$$

$$\int_{-\infty}^{\infty} e^{-p^2} p^4 dp = \frac{3\sqrt{\pi}}{4}$$

(b) Find the solution to the heat equation with initial condition $u(x,0) = x^5$.
Since $u(x,t)$ is a solution to the diffusion equation, so is $u_x(x,t)$, but

$$u_x(x,0) = 5x^4$$

Therefore, by uniqueness of solutions

$$u_x(x,t) = 5(x^4 + 12tx^2 + 12t^2)$$

To find u(x,0) we integrate with respect to x

$$u(x,t) = x^5 + 20tx^3 + 60t^2x + f(t)$$

Since $u(x,0) = x^5$, we see that $f(0) = 0$. Also

$$u_t(x,t) = 20x^3 + 120tx + f'(t)$$

$$u_{xx}(x,t) = 20x^3 + 120tx$$

So since $u_t = u_{xx}$ We have $f'(t) = 0$, and so

$$u(x,t) = x^5 + 20tx^3 + 60t^2x$$

5. Solve the wave equation with friction

$$u_{xx} = u_{tt} + 2u_t \qquad 0 < x < \pi, t > 0$$

with initial conditions

$$u(x,0) = \sin(x) \qquad u_t(x,0) = 0.$$

(Hint: Look for separated solutions.)

Set $u(x,t) = F(x)G(t)$ Then

$$u_x x = F''(x)G(t)$$

$$u_{tt} + 2u_t = F(x)G''(t) + 2F(x)G'(t)$$

Setting the two sides equal we obtain

$$\frac{F''(x)}{F(x)} = \frac{G''(t) + 2G'(t)}{G(t)}$$

Since $u(x,0) = \sin(x)$ we see that $F(x) = \sin(x)$ and $G(0) = 1$. Then we have

$$-1 = \frac{G''(t) + 2G'(t)}{G(t)}$$

or

$$G''(t) + 2G'(t) + G(t) = 0$$

So $G(t) = Ae^t + Bte^t$. Since $u_t(x,0) = 0$, we also have $G'(0) = 0$, so we have

$$1 = G(0) = A$$

$$0 = G'(0) = A + B$$

so $B = -1$ and we have $G(t) = e^t - te^t$ and thus

$$u(x,t) = \sin(x)(e^t - te^t)$$

Check

$$u_{xx} = -\sin(x)(e^t - te^t)$$

$$u_{tt} + 2u_t = -\sin(x)(e^t + te^t) - 2\sin(x)(te^t) = -\sin(x)(e^t - te^t)$$

6. Find the eigenvalues and eigenfunctions for

$$u'' + \lambda u = 0$$

with boundary conditions

$$u(0) = 0 \qquad u(1) + u'(1) = 0$$

If $\lambda > 0$ then the solutions are

$$u(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

$u(0) = 0$ implies that $A = 0$. Then

$$u(1) = \sin(\sqrt{\lambda})$$

and

$$u'(1) = \sqrt{\lambda}\cos(\sqrt{\lambda})$$

so we have

$$\sin(\sqrt{\lambda}) = -\cos(\sqrt{\lambda})$$

or

$$\tan(\sqrt{\lambda}) = -\sqrt{\lambda}$$

Thus the eigenvalues are

$$\omega_i^2 \qquad i = 1, 2, 3\ldots$$

where ω_i are the positive roots to the equation

$$\tan(\omega_i) = -\omega_i$$

And the eigenfunctions are $\sin(\omega_i x)$.
If $\lambda = 0$ the solutions are

$$u(x) = Ax + b$$

$u(0) = 0$ implies that $b = 0$,

$$u(1) = A$$

and

$$u'(1) = A$$

so $u(1) = -u'(1)$ implies that $A = 0$, and 0 is not an eigenvalue.
If $\lambda < 0$ then the solutions are

$$u(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

u(0) = 0 implies

$$B = -A$$

We then have

$$u(1) = A(e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}})$$

and

$$u'(1) = A\sqrt{-\lambda}(e^{\sqrt{-\lambda}} + e^{-\sqrt{-\lambda}})$$

$u(1) = -u'(1)$ implies that either $A = 0$ or

$$e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}} = -\sqrt{-\lambda}(e^{\sqrt{-\lambda}} + e^{-\sqrt{-\lambda}})$$

Which is impossible since the left hand side is positive and the right hand side negative, so there are no negative eigenvalues.

7. Let $f(x) = x(1-x)$ $0 \leq x \leq 1$.

   (a) Find the fourier sine series of f on $[0, 1]$.

   $$A_n = 2\int_0^1 x(1-x)\sin(n\pi x)dx$$

   integrate by parts with $u = x - x^2$ and $dv = \sin(n\pi x)$.

   $$\begin{aligned} A_n &= 2\left[-\frac{1}{n\pi}(x-x^2)\cos(n\pi x)|_{x=0}^{x=1} + \frac{1}{n\pi}\int_0^1 (1-2x)\cos(n\pi x)dx\right] \\ &= \frac{2}{n\pi}\int_0^1 (1-2x)\cos(n\pi x)dx \\ &= \frac{2}{n\pi}\left[\frac{1}{n\pi}(1-2x)\sin(n\pi x)|_{x=0}^{x=1} + \frac{2}{n\pi}\int_0^1 \sin(n\pi x)dx\right] \\ &= \frac{4}{n^2\pi^2}\int_0^1 \sin(n\pi x)dx \\ &= \frac{-4}{n^3\pi^3}(\cos(n\pi) - 1) \end{aligned}$$

   Where we have integrated by parts again in going from the second to third lines.
   So we have that the coefficients are zero if n is even and $\frac{8}{n^3\pi^3}$ when n is odd. Thus the Fourier sine series is

   $$\frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi x)}{(2n-1)^3}$$

   (b) What is the solution to the wave equation

   $$u_{tt} = u_{xx}$$

with Dirichlet boundary conditions

$$u(0,t) = u(1,t)$$

and initial conditions

$$u(x,0) = f(x)$$

$$u_t(x,0) = 0$$

The solution is

$$u(x,t) = \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{\cos((2n-1)\pi t)\sin((2n-1)\pi x)}{(2n-1)^3}$$