Math 425 - Midterm exam - March 5, 2009

NAME:..

1. ....

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Total Points ....

1. Solve the initial value problem

$$u_{tt} + u_{xt} - 2u_{xx} = 0$$
$$u(x,0) = x^2$$
$$u_t(x,0) = 0$$

Method 1:

We write

$$u_{tt} + u_{xt} - 2u_{xx} = \left(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u = 0.$$

Let

$$v = u_t - u_x.$$

Then our equation becomes

$$v_t + 2v_x = 0.$$

The solution is

$$v(x,t) = f(x - 2t).$$

We then have

$$u_t - u_x = f(x - 2t).$$

We first solve the homogeneous equation

$$u_t^h - u_x^h = 0.$$

The solution ia $u^h(x,t) = g(x+t)$.

We now must just find a particular solution. We guess this solution should be of the form

$$u^p(x,t) = h(x - 2t)$$

Then

$$u_t^p - u^p x = h'(x-2t) - 2h'(x-2t) = 3h'(x-2t) = f(x-2t)$$

So if we take $h(w) = \frac{1}{3}\int_0^w f(p)dp$ we see that $h(x-2t)$ will be a solution.

Therefore adding our homogeneous and particular solution we have the general solution

$$u(x,t) = u^p(x,t) + u^h(x,t) = g(x+t) + h(x-2t)$$

where g and h are arbitrary functions.

To find g and h we plug in the initial conditions

$$x^2 = g(x) + h(x)$$

$$0 = g'(x) - 2h'(x)$$

Differentiating the first equation wrt x gives

$$2x = g'(x) + h'(x)$$

Subtracting the first equation from the third gives us

$$2x = 3h'(x)$$

So $h(x) = \frac{1}{3}x^2 + c_1$. On the other hand, multiplying the third equation by 2 and adding it to the second equation gives us

$$4x = 3g'(x)$$

So $g(x) = \frac{2}{3}x^2 + c_2$

Then $u(x,t) = \frac{1}{3}(x-2t)^2 + \frac{2}{3}(x+t)^2 + c_1 + c_2$.

Plugging in $u(x,0) = x^2$ we see that $c_1 + c_2 = 0$.

The equation can be simplified

$$\begin{aligned}
\frac{1}{3}(x-2t)^2 + \frac{2}{3}(x+t)^2 &= \frac{1}{3}\left(x^2 - 4xt + 4t^2\right) + \frac{2}{3}\left(x^2 + 2xt + t^2\right) \\
&= \frac{x^2 - 4xt + 4t^2 + 2x^2 + 4xt + 2t^2}{3} \\
&= \frac{3x^2 + 6t^2}{3} \\
&= x^2 + 2t^2
\end{aligned}$$

Method 2:

Set $v = u_x$, then v is the solution to

$$v_{tt} + v_{xt} - 2v_{xx} = 0$$

$$v(x,0) = 2x$$

$$v_t(x,0) = 0$$

But just by looking at the equation we can see that $v(x,t) = 2x$ is the solution. We then have

$$u_x(x,t) = 2x$$

Integrating with respect to x gives

$$u(x,t) = x^2 + f(t)$$

Where $f(t)$ is some function of t. From the initial conditions we have

$$f(0) = 0$$

$$f'(0) = 0$$

and

$$f''(t) = 4$$

So we have $f(t) = 2t^2$, so our solution is

$$u(x,t) = x^2 + 2t^2$$

2. Consider the equation

$$u_t + u = u_{xx} \tag{1}$$

(a) Suppose u is a solution to (1) such that

$$u(x, 0) \leq C$$

$$u(0, t) \leq Ce^{-t}$$
$$u(1, t) \leq Ce^{-t}$$

for a fixed constant C.
Show that $u(x, t) \leq Ce^{-t}$ for all x and t in the rectangle

$$0 \leq x \leq 1 \qquad 0 \leq t \leq T$$

Solution:
Set $v(x, t) = u(x, t)e^t$, then

$$\begin{aligned} v_t &= u_t e^t + ue^t \\ &= (u_t + u)e^t \\ &= u_{xx} e^t \\ &= v_{xx} \end{aligned}$$

So $v(x, t)$ is a solution to the diffusion equation. By the maximum principle the maximum of v on the rectangle

$$0 \leq x \leq 1 \qquad 0 \leq t \leq T$$

must occur on the bottom or the sides.
On the bottom of the rectangle

$$v(x, 0) = u(x, 0)e^0 \leq C$$

and along the sides

$$v(0, t) = u(0, t)e^t \leq C$$
$$v(1, t) = u(1, t)e^t \leq C$$

Therefore, since $v(x, t) \leq C$ on the bottom and sides of the rectangle, $v(x, t) \leq C$ on the whole rectangle.
But

$$u(x, t)e^t \leq C$$

so

$$u(x, t) \leq Ce^{-t}$$

(b) Show that any solution (if it exists) to the problem consisting of (1) with Dirichlet boundary conditions

$$u(0,t) = 0 \qquad u(1,t) = 0$$

and initial condition

$$u(x,0) = \phi(x)$$

is unique.

Solution Let u_1 and u_2 be two solutions and let $w = u_1 - u_2$. Then by linearity of the equation we have that w is a solution to

$$w_t + w = w_{xx}$$

satisfying Dirichlet boundary conditions

$$w(0,t) = 0 \qquad w(1,t) = 0$$

and initial condition

$$w(x,0) = 0$$

Let $v(x,t) = w(x,t)e^t$, by the calculation done in the solution to part (a) we have that $v(x,t)$ is a solution to the diffusion equation, satisfying the Dirichlet boundary conditions which initially is

$$v(x,0) = 0$$

Therefore, by uniqueness of the solutions to the diffusion equation with Dirichlet boundary conditions we have that

$$v(x,t) = 0$$

for all x and t. This then implies $w(x,t) = 0$, in other words $u_1 = u_2$ and so the solution is unique.

3. Consider the diffusion equation on the whole line with the usual initial condition

$$u(x,0) = \phi(x)$$

Show that if ϕ is an even function then $u(x,t)$ is an even function of x for all t.

Method 1: The formula for the solution to the diffusion equation on the whole line is

$$u(x,t) = \frac{1}{4\pi kt}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)dy$$

So

$$u(-x,t) = \frac{1}{4\pi kt}\int_{-\infty}^{\infty} e^{-(x+y)^2/4kt}\phi(y)dy$$

We must show $u(x,t) = u(-x,t)$.

To see this we make the substitution $w = -y$ into the expression for $u(x,t)$. Then $dw = -dy$ and, since ϕ is even, $\phi(y) = \phi(-y) = \phi(w)$ so we have

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi kt}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)dy \\ &= \frac{1}{4\pi kt}\int_{\infty}^{-\infty} e^{-(x+w)^2/4kt}\phi(w)(-dw) \\ &= \frac{1}{4\pi kt}\int_{-\infty}^{\infty} e^{-(x+w)^2/4kt}\phi(w)dw \\ &= u(-x,t) \end{aligned}$$

Where we have replaced the dummy variable y with w in the expression for $u(-x,t)$.

Method2:

Set $v(x,t) = u(x,t) - u(-x,t)$.

$$\begin{aligned} v_t(x,t) &= u_t(x,t) - u_t(-x,t) \\ &= u_{xx}(x,t) - u_{xx}(-x,t) \\ &= v_{xx}(x,t) \end{aligned}$$

(because $\frac{\partial^2}{\partial x^2}(u(-x,t))) = u_{xx}(-x,t)$.)

So $v(x,t)$ is a solution to the diffusion equation on the whole line. with initial condition

$$v(x,0) = \phi(x) - \phi(-x) = 0$$

since ϕ is even. Then, by uniqueness of solutions to the diffusion equation, $v(x,t) = 0$ for all x and t, in other words

$$u(x,t) = u(-x,t)$$

4. Find the solution to the equation

$$u_{tt} - 4\pi^2 u = u_{xx}$$

with Neumann boundary conditions

$$u_x(0,t) = 0 \qquad u_x(1,t) = 0$$

and initial conditions

$$u(x,0) = 0$$
$$u_t(x,0) = \cos(2\pi x)$$

Solution: We look for a separated solution $u(x,t) = F(x)G(t)$.

First let's find what the initial conditions imply about F and G.

Since $F(x)G(0) = u(x,0) = 0$ and $F(x)G'(0) = u_t(x,0) = \cos(2\pi x)$ we have

$$G(0) = 0 \qquad G'(0) = 1 \qquad F(x) = \cos(2\pi x).$$

Note that with this choice of F, the boundary conditions are obviously satisfied.

Now plugging into the equation we have

$$F(x)G''(t) - 4\pi^2 F(x)G(t) = F''(x)G(t)$$
$$\frac{G''(t) - 4\pi^2 G(t)}{G(t)} = \frac{F''(x)}{F(x)} = \lambda$$

Since $F(x) = \cos(2\pi x)$, $\lambda = -4\pi^2$. So the equation for $G(t)$ is

$$\frac{G''(t) - 4\pi^2 G(t)}{G(t)} = -4\pi^2$$

$$G''(t) = 0$$

Since $G(0) = 0$ and $G'(0) = 1$, $G(t) = t$ and so our final solution is

$$u(x,t) = t\cos(2\pi x)$$

5. Find the cosine series of the function $x - \cos(2x)$ on the interval $0 \le x \le \pi$.

Solution:

The cosine series on the interval $0 \le x \le \pi$ is the unique way to write the function as a linear combination of the functions

$$1, \cos(x), \cos(2x), \cos(3x), \ldots$$

Therefore, the cosine series of $x - \cos(x)$ is the just the cosine series of x minus $\cos(2x)$.

$$A_0 = \frac{2}{\pi} \int_0^\pi x dx = \pi$$

$n \ge 1$

$$\begin{aligned} A_n &= \frac{2}{\pi} \int_0^\pi x \cos(nx) dx \\ &= \frac{2}{\pi} \left(\frac{x}{n} \sin(nx)|_{x=0}^{x=\pi} - \frac{1}{n} \int_0^\pi \sin(nx) dx \right) \\ &= \frac{2}{\pi n^2} (\cos(n\pi) - 1) \\ &= \frac{2}{\pi n^2} ((-1)^n - 1) \end{aligned}$$

Where in the first line we integrated by parts.

So we have, $A_n = 0$ if n is even and $A_n = \frac{-4}{\pi n^2}$ if n is odd.

So the Fourier cosine series for x is

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)x)$$

And thus the cosine series for $x - \cos(2x)$ is

$$\frac{\pi}{2} - \cos(2x) - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)x)$$