Math 661 – Spring 2009 – Homework 6

(1) Given $p, q \in M$ the excess function is

$$e_{p,q}(x) = d(p,x) + d(q,x) - d(p,q)$$

Show that $e_{p,q} \geq 0$ and that $e_{p,q} = 0$ for any point on a minimal geodesic between p and q. Use this fact and the Laplacian comparison theorem to give a different proof of the Myer's theorem, if M is complete with $\mathrm{Ric} \geq (n-1)k$ then $\mathrm{diam}_M \leq \frac{\pi}{\sqrt{k}}$.

(2) Given $f : M \to \mathbb{R}$ there is a generalization of Ricci curvature due the Bakry and Emery which gives a volume comparison theorem for the measure $e^{-f} dvol$. This is the (∞)-Bakry-Emery tensor;

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f$$

(a) Show by way of an example that there are complete noncompact manifolds with $\mathrm{Ric}_f \geq \lambda > 0$ (for some choice of f). (Hint: consider flat Euclidean space.)

Thus strictly speaking, Myer's theorem is not true for the Bakry-Emery tensor. However the weaker topological consequence of Myer's theorem, that M has finite fundamental group is still true for the Bakry-Emery tensor. Define the f-Laplacian as

$$\Delta_f(u) = \Delta(u) - \langle \nabla f, \nabla u \rangle$$

(b) For any $u : M \to \mathbb{R}$ prove the following version of the Bochner formula for the Bakry-Emery tensor and f-Laplacian.

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess} u|^2 + \mathrm{Ric}_f(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta_f(u) \rangle$$

(c) Use the Bochner formula to show that if $\mathrm{Ric}_f \geq \lambda$ then for any geodesic γ and $t > 1$

$$\Delta_f(r)(\gamma(t)) \leq \Delta_f(\gamma(1)) - \lambda(t-1)$$

(d) Show that if M is complete and $\mathrm{Ric}_f \geq \lambda > 0$ then

$$\int_M e^{-f} dvol < \infty$$

Hint : Define $\mathcal{A}_f = e^{-f}\mathcal{A}$ and show that $\frac{\mathcal{A}'_f}{\mathcal{A}_f} = \Delta_f(r)$. Now integrate out in a similar way as we did in proving the regular volume comparison.

(e) Show that $\pi_1(M)$ is finite if M is complete and there is a function f on M such that $\mathrm{Ric}_f \geq \lambda > 0$. (Hint: Apply the volume comparison theorem to the universal cover of M with the function $\tilde{f} = f(\pi(x))$ where π is the covering map.)