Math 240                                      Midterm I                                      October 7, 2010
Professor Ziller                              

Name: ______________________________________

Signature: __________________________________

TA: ________________________________________

Recitation Day and Time: ______________________

You need to show all of your work. A correct answer with no work will get 0 points. If you see a shortcut, you need to explain it. Please circle the answer for each problem. Each problem is worth 10 points. It is more important to do the problems well that you know how to do, than it is to finish the whole exam.

(Do not fill these in; they are for grading purposes only.)

1) 

2) 

3) 

4) 

5) 

6) 

7) 

8) 

9) 

10) ___________________________

Total
1. Find the solution of \( y'' - 3y' + 2y = 10 \sin(x) \) with \( y(0) = 1 \), \( y'(0) = 0 \).

\[
\begin{align*}
\mu(x) & = x^2 - 3x + 2 = (x-2)(x-1) = 0 \\
g_1 & = 2 \\
g_2 & = 1 \\
y_1 & = c_1 e^{2x} + c_2 e^x \\
y_p & = A \sin x + B \cos x \\
y_p' & = A \cos x - B \sin x \\
y_p'' & = -A \sin x - B \cos x \\
y'' - 3y' + 2y & = \cos x (-B - 3A + 2B) + \sin x (-A + 3B + 2A) \\
& = 10 \sin x \\
B - 3A & = 0 \\
3B + A & = 10 \\
B & = 3 \\
A & = 1 \\
y_p & = 3 \sin x + 3 \cos x \\
y & = c_1 e^{2x} + c_2 e^x + 3 \sin x + 3 \cos x \\
y(0) & = c_1 + c_2 + 3 = 1 \\
\therefore c_1 + c_2 & = -2 \\
y'(0) & = 2c_1 + c_2 + 1 = 0 \\
\therefore 2c_1 + c_2 & = -1 \\
c_1 & = 1 \\
c_2 & = -3 \\
\boxed{y = e^{-3} e^x + 3 \sin x + 3 \cos x}
\end{align*}
\]
2. Consider the linear system of equations $Ax = b$ and let $M$ be the row echelon form of the augmented matrix $(A | b)$. In each of the cases below,

(a) Bring the augmented matrix $M$ into its reduced row echelon form.

(b) Identify the set of solutions of $Ax = b$ as the empty set, a point, a line, or a 3-space.

(I) \[
\begin{pmatrix}
1 & 0 & 1 & 3 & 2 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]  

(II) \[
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 1 & 2
\end{pmatrix}
\]

1. \[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 2 & 2 & 1 \\
0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

$x = 1 - w$

$y = t$

$z = 1 - 2s$

$x = 1 - s$

$solution depends on 2 parameters - 2 plane in 4 space$

2. \[
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 \\
0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 2 & 0
\end{pmatrix}
\]

$-1 \times \begin{pmatrix}
1 & 2 & 0 & 1 & -1 \\
0 & 0 & 1 & 1 & 2 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}$

$-1 \times \begin{pmatrix}
1 & 2 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}$

$x = -1 - 2y$

$z = 2$

$w = 0$

$y = s$

$x = -1 - 2s$

$x = t$

$w = 0$

$depends on one parameter - line in 4 space$
3. Find a second order linear homogeneous differential equation where $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ are solutions.

$r = 2$ must be a double root

aux eqn $\quad (r-2)^2 = 0 \quad \Rightarrow \quad r^2 - 4r + 4 = 0$

diff eqn must be $\quad y'' - 4y' + 4y = 0$
4. Consider the following system of equations:
\[
\begin{align*}
x + y + 2z &= 0 \\
x + 2y + 5z &= 0 \\
x + y + (k + 3)z &= 0
\end{align*}
\]

(a) For what values of \( k \) does there exist a non-zero solution?

(b) Let \( k \) be the value determined in (a). On how many arbitrary parameters do the solutions depend?

\[
\begin{pmatrix}
1 & 1 & 2 \\
1 & 2 & 5 \\
1 & 1 & (k+3)
\end{pmatrix}
\begin{pmatrix} x \\ y \\ z \end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

If \( (k+1) \neq 0 \), then the solution is unique. In fact, \( x = y = z = 0 \).

If \( k \neq -1 \), then

\[
\begin{pmatrix}
0 & 1 & 2 \\
0 & 1 & 3 \\
0 & 0 & (k+1)
\end{pmatrix}
\begin{pmatrix} x \\ y \\ z \end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 \\ 0 \\ -1 \\
0 & 1 & 3 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} x \\ y \\ z \end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

\( z = 5 \), \( x = 5 \), \( y = -3s \) depends on 1 parameter.
5. A spring satisfies the differential equation $y'' + 4y = 0$ with initial conditions $y(0) = \sqrt{2}$, $y'(0) = 2\sqrt{2}$.

(a) What is the amplitude and period of the motion?

(b) At what time does it reach its equilibrium position for the first time?

(b) Draw the graph of the motion.

\[ y^2 + 4 = 0 \quad r = \pm 2i \]

\[ y = C_1 \cos(2t) + C_2 \sin(2t) \]

\[ y(0) = C_1 = \sqrt{2} \quad y'(0) = 2C_2 = 2\sqrt{2} \]

\[ y = \sqrt{2} \cos(2t) + \sqrt{2} \sin(2t) \]

\[ \sqrt{A^2 + B^2} = \sqrt{4} = 2 \]

\[ \sin \delta = \frac{B}{2} = \frac{1}{\sqrt{2}} \quad \cos \delta = \frac{A}{2} = \frac{1}{\sqrt{2}} \]

\[ y = 2 \sin(\sqrt{2}t + \frac{\pi}{4}) = 2 \cos(2t - \frac{\pi}{4}) \]

(a) Amplitude = 2 \quad Period = \frac{2\pi}{2} = \pi

b) $2t + \frac{\pi}{4} = \pi$ or $t = \frac{3\pi}{8}$

c1 \[ \text{Graph showing the motion.} \]
6. Find the solution of \( y'' = \frac{2y}{x^2} \) with \( y(1) = 1 \) and \( y'(1) = 2 \).

\[
X^2 y'' - 2y = 0 \quad \text{Cauchy Euler}
\]

\[
\tau (\tau - 1) - 2 = \tau^2 - \tau - 2 = (\tau - 2)(\tau + 1) = 0
\]

\( \tau_1 = 2 \quad \tau_2 = -1 \)

\[
y = c_1 x^2 + c_2 \frac{1}{x}
\]

\[
y(1) = c_1 + c_2 = 1
\]

\[
y'(1) = 2c_1 - c_2 = 2
\]

\[
y = x^2
\]
7. Let \( y(x) \) be the solution of \( y'' = x + y \) that passes through the origin and has a horizontal tangent line there. What is the value of \( \lim_{x \to \infty} \frac{y(x)}{e^x} \)?

\[
y'' - y = x
\]

\[
y = e^{rx}
\]

\[
y = c_1 e^x + c_2 e^{-x}
\]

\[
y' = Ay
\]

\[
y'' - y = -Ax - B = x
\]

\[
y = \frac{1}{2} e^x - \frac{1}{2} e^{-x} - x
\]

\[
y = \frac{1}{2} e^x - \frac{1}{2} e^{-x} - x
\]

\[
\lim_{x \to \infty} \frac{y}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0
\]

\[
\lim_{x \to 0} \frac{y}{e^x} = \frac{1}{2}
\]
8. Find the general solution of \( y'' = 4y' - 4y + e^{2x} \).

\[
\begin{align*}
\gamma'' - 4\gamma' + 4\gamma &= e^{2x} \\
\gamma'' - 4\gamma' + 4 &= (\tau - 2)^2 = 0 \\
\gamma_h &= c_1 e^{\tau x} + c_2 xe^{\tau x} \\
\gamma_p &= \lambda x^2 e^{2x} \quad \text{since } e^{2x} \text{ and } xe^{2x} \text{ solve homogeneous eqn.} \\
\gamma' &= 2\lambda xe^{2x} + 2\lambda x^2 e^{2x} \\
\gamma'' &= 2\lambda e^{2x} + 4\lambda xe^{2x} + 4\lambda x^2 e^{2x} + 4\lambda x^2 e^{2x} \\
\gamma'' - 4\gamma' + 4\gamma &= 2\lambda e^{2x} (4A - 8A + 4A) \\
&+ xe^{2x} (8A - 8A) \\
&+ 2\lambda e^{2x} = 2\lambda e^{2x} = e^{2x} \quad A = \frac{1}{2} \\
\gamma &= c_1 e^{2x} + c_2 xe^{2x} + \frac{1}{2} x^2 e^{2x}
\end{align*}
\]
9. You are given that \( y = x^{-1}e^x \) is a particular solution of the inhomogeneous differential equation

\[
y'' - 2y' + y = \frac{2}{x^3}e^x
\]

Find the solution with \( y(1) = e, \ y'(1) = 1 \).

\[
y_h \quad y'' - 2y' + y = 0
\]

\[
\begin{align*}
    a_n &= r^2 - 2r + 1 = (r - 1)^2 \\
    r_1 &= r_2 = 1
\end{align*}
\]

\[
y_p = \frac{1}{x} e^x
\]

\[
y = y_h + y_p = c_1 e^x + c_2 x e^x + \frac{1}{x} e^x
\]

\[
y(1) = c_1 e + c_2 e + e = e \quad c_1 + c_2 = e
\]

\[
y'(1) = c_1 e + c_2 e + c_2 e - e + e = c_1 e + 2c_2 e = 1
\]

\[
c_1 + c_2 = 0 \quad c_1 = -c_2
\]

\[
c_1 + 2c_2 = \frac{1}{e} \quad c_2 = \frac{1}{e} \quad c_1 = -\frac{1}{e}
\]

\[
y = -\frac{1}{e} e^x + \frac{1}{e} x e^x + \frac{1}{x} e^x
\]
10. This is the only problem where no work needs to be shown. Each answer is worth one point.

Which one of the following statements are true or false:

(a) The solution $y(t)$ of a harmonic motion is periodic, i.e $y(t + L) = y(t)$ for some $L$ and all $t$. True ☒ False ☐
(b) The solution $y(t)$ of a damped harmonic motion is periodic, i.e $y(t + L) = y(t)$ for some $L$ and all $t$. True ☐ False ☒
(c) If $y_1$ and $y_2$ are solutions of an inhomogeneous linear differential equation, then $y_1 - y_2$ is a solution of the corresponding homogeneous differential equation. True ☒ False ☐
(d) If $y(t)$ is the solution of a critically damped harmonic motion, then $y = 0$ has a unique solution. True ☐ False ☒
(e) For any $m \times n$ matrix $A$, the matrix $AA^T$ is symmetric. True ☒ False ☐
(f) A homogeneous system $Ax = 0$ of 3 equations in 4 unknowns always has non-zero solutions. True ☐ False ☒
(g) For a homogeneous differential equation $y'' + cy' + ky = 0$, with $c > 0$, $\lim_{t \to \infty} y(t) = \infty$ is not possible. True ☒ False ☐
(h) If $A$ is a $2 \times 2$ matrix with $A^2 = 0$, then $A = 0$. True ☐ False ☒
(i) If $y(t)$ is the solution of a damped harmonic motion, then $\lim_{t \to \infty} y(t) = 0$. True ☐ False ☒
(j) If $y'' + 9y = \sin(\alpha t)$ is the differential equation of a forced spring, then the value of $\alpha$ under which one has resonance is 9. True ☐ False ☒
11. Extra Credit Problem.

You should not attempt this problem unless you were able to finish all other problems to your own satisfaction. The score on this problem will not affect the curve, but will be used as an extra consideration in assigning your grade at the end of the semester. There is no partial credit on this problem whatsoever!

A spring satisfies the differential equation $4y'' + ky' + y = 0$.

If the spring is critically damped and a weight is attached to the spring is released 2 m above the equilibrium position, at a downward speed of $A$ m/sec (assuming that $A > 0$). Determine all the possible values of $A$ for which $\lim_{t \to \infty} y(t) = 0$, but the weight never reaches the equilibrium position.

a) $4r^2 + kr + 1 = 0 \quad r = \frac{-k \pm \sqrt{k^2 - 16}}{8}$

Critically damped $k = 16$  Underdamped $k^2 < 16$

$k = 4$

b) $k = 4 \quad r = \frac{-k}{8} = -\frac{1}{2}$

$y(0) = -2 \quad y'(0) = A$

$y = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t}$

$y'(0) = -\frac{1}{2} c_1 + c_2 = A$

$1 + c_2 = A \quad c_2 = A - 1$

$y = -2 e^{-\frac{1}{2}t} + (A - 1) t e^{-\frac{1}{2}t}$  $\lim_{t \to \infty} y(t) = 0$ for all $A$

$y$ crosses equilibrium if $y = 0$

or $(A - 1) t e^{-\frac{1}{2}t} = 2 e^{-\frac{1}{2}t}$  or $t = \frac{2}{A - 1}$

But need $t > 0$ i.e. if $A > 1$ it crosses equiv.

if $A \leq 1$ it never reaches equiv.