ASSIGNMENT # 6
MATH 660, DIFFERENTIAL GEOMETRY

These exercises explore the geometry of \( \mathbb{C}P^n \). We use the following notation: \( S^1 \) acts via the Hopf action on \( S^{2n+1}(1) \subset \mathbb{C}^{n+1} \) as \( x \rightarrow e^{i\theta}x \) for \( x \in \mathbb{C}^{n+1} \). Then \( \mathbb{C}P^n = S^{2n+1}/S^1 \) with its induced Fubini Study metric. \([x] \in \mathbb{C}P^n\) denotes the equivalence class \( x \simeq e^{i\theta}x \) and the tangent space \( T_{[x]}\mathbb{C}P^n \) is represented by \( ([x,v]) \) with \( v \in H_x \), i.e. \( \langle v, x \rangle = \langle v, ix \rangle = 0 \), and equivalence relationship \( ([x,v]) \simeq ([e^{i\theta}x, e^{i\theta}v]) \).

(1) Show that \( \mathbb{C}P^1 \) with its Fubini-Study metric is isometric to \( S^2(\frac{1}{2}) \).

(2) Compute the full isometry group of the Fubini Study metric on \( \mathbb{C}P^n \).

   Hint: Use the fact that isometries preserve sectional curvature.

(3) Show that the only isometric quotient of \( \mathbb{C}P^n \) is, up to isometry, obtained when \( n = 2k - 1 \) and by dividing by the involution

\[
[z_0, z_1, z_2, z_3, \ldots, z_{2k-1}, z_{2k}] \rightarrow [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \ldots, \bar{z}_{2k}, -\bar{z}_{2k}].
\]

(4) Show that \( J([x,v]) = [x,iv] \) for \( [x,v] \in T_{[x]}\mathbb{C}P^n \) is a well defined complex structure and that \( \nabla J = 0 \), i.e. the Fubini Study metric is Kähler.

(5) Let \( M \) be \( \mathbb{C}P^n \) with its Fubini Study metric.

   (a) A linear subspace \( V \subset T_{[x]}\mathbb{C}P^n \) is called complex if \( JV = V \) and totally real if \( JV \perp V \). Show that if \( V \) is complex, \( \exp_{[x]}(V) \) is totally geodesic and isometric to \( \mathbb{C}P^k \subset \mathbb{C}P^n \) with its Fubini Study metric. Show that if \( V \) is totally real, \( \exp_{[x]}(V) \) is totally geodesic and isometric \( \mathbb{R}P^k(1) \subset \mathbb{R}P^n(1) \subset \mathbb{C}P^n \).

   (b) Show that these two types of submanifolds are the only ones which are totally geodesic.

   Hint: Use (and prove) the fact that for a totally geodesic submanifold \( N \subset M \) we have \( R(u,v)w \subset T_pN \) if \( u, v, w \in T_pN \). You need to derive a formula for \( R(u,v)v \) for the Fubini Study metric.