ASSIGNMENT # 8
MATH 660, DIFFERENTIAL GEOMETRY

(1) Let $N_i \subset M^n$ be two submanifolds of dimension $n_i$.
   (a) If $N_1 \cap N_2 = \emptyset$, and $c : [0, a] \to M$ is a shortest connection from $N_1$ to $N_2$, show that $c$ is a geodesic that meets the submanifolds $N_i$ orthogonally at the endpoints.
   (b) Assume that $\gamma$ is a geodesic from $N_1$ to $N_2$ that meets the submanifolds orthogonally. Develop a second variation formula for variational vector fields $V$ along $c$ which are tangent to $N_i$ at the endpoints.
   (c) Show that every such vector field $V$ comes from a variation of curves $c_s$ which all start on $N_1$ and end on $N_2$.

(2) Let $M^n$ be a complete Riemannian manifold.
   (a) If $M$ has positive sectional curvature, and $N_i$ are 2 compact totally geodesic submanifolds with $n_1 + n_2 \geq n$, show that $N_1$ and $N_2$ must intersect.
   (b) Give an example that the dimension assumption in (a) is necessary.
   (c) Show that 2 compact minimal hypersurfaces in a manifold with positive Ricci curvature must intersect.

(3) Fix $p \in M$ and for each unit vector $v \in T_p M$ let $t(v)$ be the first conjugate point along the geodesic $\exp(tv)$. Show that $t$ is continuous.
   Hint: Show it is upper and lower continuous, and for one case use the index form.

(4) Let $M$ be a complete simply connected manifold with non-positive sectional curvature.
   (a) In class we showed that $|d(\exp)_p)_v(w)| \geq |w|$ for all $v, w \in T_p M$. If we have equality, show that $\sec(\gamma'(t), E_w) = 0$ for all $t \leq 1$, where $\gamma(t) = \exp(tv)$ and $E_w$ is the parallel vector field along $\gamma$ with $E_w(0) = w$. Furthermore, $E_w$ is also a Jacobi field.
   (b) Assume you have a geodesic triangle defined by 3 geodesics $\gamma_i$, $i = 1, 2, 3$ with angles at the vertices given by $\alpha_i$, $i = 1, 2, 3$. In class we showed that $\sum \alpha_i \leq \pi$. If you have equality, show that the triangle is the boundary of a flat totally geodesic surface.
   Hint for (a): If $A$ is symmetric endomorphism with $\langle Av, v \rangle \geq 0$ for all $v$, and $v_0$ a vector with $\langle Av_0, v_0 \rangle = 0$, show that $Av_0 = 0$.
   Hint for (b): The proof in class was one vertex and angle at a time. Discuss equality at one vertex first, and then use a second vertex as well.

(5) Let $M$ be a simply connected complete Riemannian manifold with non-positive sectional curvature.
   (a) If $\gamma$ is a geodesic and $p$ a point not on $\gamma$, show that $f(t) = d^2(p, \gamma(t))$ is a strictly convex function, and that there is a unique point $\gamma(t_0)$ closest to $p$.
   (b) Show that for all $p \in M$ and $r > 0$, the ball $B_r(p)$ is strictly convex, i.e. any geodesic connecting 2 points in $B$, completely lies in $B$. 