ASSIGNMENT # 9
MATH 660, DIFFERENTIAL GEOMETRY

(1) Let \( \gamma: [0, a] \to M \) be a geodesic and \( L(c) = \int_0^a |c'(t)|dt \) be the length of a curve. Let \( c_s \) be a proper variation (i.e. end points fixed) with \( c_0 = \gamma, V = \frac{d(c_s)}{ds} |_{s=0} \) its variational vector field, and \( L(s) = L(c_s) \). Show that \( L'(0) = 0 \) and
\[
L''(0) = \int_0^a \langle V'_\perp, V'_\perp \rangle - \langle R(V\perp, \gamma')\gamma', V\perp \rangle dt
\]
where \( V\perp = V - \langle V, \gamma' \rangle \gamma'/|\gamma'|^2 \) is the component of \( V \) orthogonal to \( \gamma' \).

(2) Prove the Morse Schönberg comparison theorem:
Let \( \gamma: [0, a] \to M^n \) be a normal geodesic and \( \sec \geq \delta \). Show that if \( L(\gamma) = a > k\sqrt{\frac{n}{\delta}} \), for some positive integer \( k \), then the index of \( \gamma \) satisfies \( \text{ind}(\gamma) > k(n - 1) \).
Thus, by the index theorem, \( \gamma \) has at least \( k(n - 1) \) conjugate points (counted with multiplicity). What is the corresponding statement for \( \sec \leq \delta \)?
Hint: Compare the index form of \( \gamma \) with the index form of a geodesic \( \bar{\gamma} \) in a space of constant curvature \( \delta \).

(3) Let \( N^k \subset M^n \) be a submanifold and \( \gamma: [0, a] \to M \) a geodesic with \( \gamma(0) \in N, \gamma'(0) \) orthogonal to \( N \), and \( \gamma(a) = p \).
(a) If \( \gamma \) has no focal points for \( t \leq a \), show that there exists a neighborhood \( U \) of \( \gamma \) such that any curve from \( N \) to \( p \), which is completely contained in \( U \), is longer than \( \gamma \) unless it is a reparametrization of \( \gamma \).
(b) If there exists a focal point \( \gamma(t_0) \) with \( t_0 < a \), show that there exist nearby curves from \( N \) to \( p \) which are shorter than \( \gamma \). (You need to use the formula for the index form \( I \) in problem (3) below).
Hint: For part (a) you need to use the Gauss Lemma from problem (1) in the next Assignment # 10.

(4) Recall the following: Let \( \gamma \) be a geodesic as in problem (3) above and define the index form as
\[
I(V, W) = \int_0^a \langle V', W' \rangle - \langle R(V, \gamma')\gamma', W \rangle dt + \langle S_{\gamma'}(V), W \rangle_{t=0}.
\]
Then for any variation \( c_s \) from \( N \) to \( p \) with variational vector field \( V \), \( E''(0) = I(V, V) \). Show that \( I \) is positive definite iff \( \gamma \) has no focal points. (You need to first determine the null space of \( I \).)

(5) Let \( N \subset \mathbb{R}^3 \) be a paraboloid \( z = y^2 \). Determine the focal points along any geodesic starting normal to \( N \). Up to what value of \( t \) are the parallel hypersurfaces \( N_t = \{ \exp(tn_p) \mid p \in N \} \) smooth (where \( n \) is the inward pointing normal).

(6) Show that if \( M \) has sectional curvature \( \sec \leq 0 \), then a geodesic in \( M \) has no focal points.