

SEVEN DIMENSIONAL COHOMOGENEITY ONE MANIFOLDS WITH NONNEGATIVE CURVATURE

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ABSTRACT. We show that a certain family of cohomogeneity one manifolds does not admit an invariant metric of nonnegative sectional curvature, unless it admits one with positive curvature. As a consequence, the classification of nonnegatively curved cohomogeneity one manifolds in dimension 7 is reduced to only one further family of candidates

A group action is called a cohomogeneity one action if its generic orbits are hypersurfaces. Such actions have been used frequently to construct examples of various types, Einstein metrics, soliton metrics, metrics with positive or non-negatively curvature, and metrics with special holonomy, see e.g. [DW, FH, GKS, GZ, GVZ, KS] for a selection of such results. The advantage of such metrics is that geometric problems are reduced to studying its behavior along a fixed geodesic $c(t)$ normal to all orbits.

In [Ho] one finds a classification of cohomogeneity one actions on n -dimensional simply connected compact manifolds of dimensions $n \leq 7$. As a consequence, he showed that such manifolds admit an invariant metric of nonnegative curvature, unless it is a member of a certain family of Brieskorn varieties B_d , or a member of two 7-dimensional families $E_{p,q}$ and $E_{p,q}^*$. In [GVWZ] it was shown that the Brieskorn varieties B_d do not admit an invariant metric of nonnegative curvature for $d \geq 3$, while $B_1 = S^7$ and $B_2 = T^1S^4$ admit nonnegatively curved invariant metrics. For the remaining two families, the questions remained open.

To describe these two families, we recall the structure of a cohomogeneity one manifold. Let G be a Lie group with subgroups K_{\pm} and $H \subset K_{\pm}$ such that K_{\pm}/H is a sphere. Then K_{\pm} acts linearly on a disc D_{\pm} and $M = G \times_{K_-} D_- \cup_{G/H} G \times_{K_+} D_+$ defines a cohomogeneity one manifold under the left action by G . In terms of this language, $E_{p,q}$ is defined by

$$G = S^3 \times S^3, \quad K_- = \{(e^{ip\theta}, e^{iq\theta}) \mid \theta \in \mathbb{R}\}, \quad K_+ = \{(a, a) \mid a \in S^3\} \cdot H, \quad \text{and } H \simeq \mathbb{Z}_2 = \{(\pm 1, 1)\}$$

where we assume that $\gcd(p, q) = 1$ and p even.

The special case of $E_{p,q}$ with $|p \pm q| = 1$ is an Eschenburg space, which admits an invariant metric with positive curvature, see [E, Zi]. In this paper we show:

THEOREM. *The cohomogeneity one manifold $E_{p,q}$ does not admit an $S^3 \times S^3$ invariant metric with nonnegative curvature if $|p \pm q| > 1$.*

The second family $E_{p,q}^*$ is determined by the same groups G and K_- , but H is trivial and K_+ is connected. The special case of $E_{1,1}^*$ is diffeomorphic to $\mathbb{C}P^2 \times S^3$, and the product metric is invariant under G and has nonnegative curvature. For the remaining manifolds, we do not know if they admit invariant metrics with nonnegative curvature.

Altogether we obtain:

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COROLLARY. *A 7-dimensional cohomogeneity one manifold admits an invariant metric with nonnegative curvature, unless it is a Brieskorn variety B_d^7 with $d > 2$, or $E_{p,q}$ with $|p \pm q| > 1$, or possibly $E_{p,q}^*$ with $(p, q) \neq (\pm 1, \pm 1)$.*

We now describe how to obtain the obstruction. Let $c(t)$ be a normal geodesic perpendicular to all orbits. It meets the singular orbits at time $t = nL$, $n \in \mathbb{Z}$ with $c(2nL) \in G/K_-$ and $c((2n+1)L) \in G/K_+$. It is sufficient to describe the metric along the geodesic c , which is determined by a collection of functions describing the homogenous metric on the principal orbits G/H . At $t = nL$, the functions must satisfy certain smoothness conditions in order for the metric to extend smoothly over the singular orbits. The Weyl group, i.e. the stabilizer of the normal geodesic, implies some of these conditions. In particular, all functions must be even at $t = (2n+1)L$. At $t = 0$ and $t = 2L$ the stabilizer groups are the same, and hence the smoothness conditions as well. But in this case they are not implied by the action of the Weyl group and we use the methods in [GVZ] to determine them. In particular, they depend on p, q and $|p \pm q| \geq 3$ implies that some functions are odd. We then use the concavity of virtual Jacobi fields developed in [VZ1] in order to show that one of the action fields Z^* , $Z \in \mathfrak{g}$ is parallel on $[0, 2L]$. This eventually leads to a contradiction by showing that the length of another action field is constant on $[0, 2L]$, but on the other hand must vanish at $t = L$ due to the smoothness conditions. For the family E_{pq}^* the smoothness conditions at $t = 0$ change and imply that all functions are even, and thus the proof breaks down. On the other hand, the method still implies strong restrictions on a possible metric with nonnegative curvature.

1. PRELIMINARIES

For a general reference for this section see e.g. [AA, Zi]. A compact cohomogeneity one manifold is the union of two homogeneous disc bundles. Given compact Lie groups H, K^-, K^+ and G with inclusions $H \subset K^\pm \subset G$ satisfying $K^\pm/H = \mathbb{S}^{\ell^\pm}$, the transitive action of K^\pm on \mathbb{S}^{ℓ^\pm} extends to a linear action on the disc $\mathbb{D}^{\ell^\pm+1}$. We can thus define $M = G \times_{K^-} \mathbb{D}^{\ell^-+1} \cup G \times_{K^+} \mathbb{D}^{\ell^++1}$ glued along the boundary $\partial(G \times_{K^\pm} \mathbb{D}^{\ell^\pm+1}) = G \times_{K^\pm} K^\pm/H = G/H$ via the identity. G acts on M on each half via left action in the first component. This action has principal isotropy group H and singular isotropy groups K^\pm . One possible description of a cohomogeneity one manifold is thus simply in terms of the Lie groups $H \subset \{K^-, K^+\} \subset G$.

A G invariant metric is determined by its restriction to a geodesic c normal to all orbits. At the points $c(t)$ which are regular with respect to the action of G , the isotropy is constant equal to H . We fix a biinvariant inner product Q on the Lie algebra \mathfrak{g} and choose a Q -orthogonal splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. At a regular point we identify the tangent space to the orbit G/H , i.e. $\dot{c}^\perp \subset T_{c(t)}M$, with \mathfrak{h}^\perp via action fields: $X \in \mathfrak{h}^\perp \rightarrow X^*(c(t))$. H acts on \mathfrak{h}^\perp via the adjoint representation and a G invariant metric on G/H is described by an $\text{Ad}(H)$ invariant inner product on \mathfrak{h}^\perp . Along c the metric on M is thus given by $g = dt^2 + g_t$ with g_t a one parameter family of Ad_H invariant metrics on \mathfrak{h}^\perp . These are described by endomorphisms

$$P_t: \mathfrak{h}^\perp \rightarrow \mathfrak{h}^\perp \quad \text{where} \quad g_t(X^*, Y^*)_{c(t)} = Q(P_t X, Y), \quad \text{for} \quad X, Y \in \mathfrak{h}^\perp.$$

These endomorphisms P_t commute with the action of Ad_H since the metric is Ad_H invariant and we thus choose a fixed Q -orthogonal splitting

$$\mathfrak{h}^\perp = \mathfrak{n}_0 \oplus \mathfrak{n}_1 \oplus \dots \oplus \mathfrak{n}_r.$$

where Ad_H acts trivially on \mathfrak{n}_0 and irreducibly on $\mathfrak{n}_i, i > 0$. The metric P_t is arbitrary on \mathfrak{n}_0 , and a multiple of Id on $\mathfrak{n}_i, i > 0$. Furthermore, \mathfrak{n}_i and \mathfrak{n}_j are orthogonal if the representations of Ad_H are inequivalent. If they are equivalent, inner products are described by 1, 2 or 4 functions, depending on whether the equivalent representations are orthogonal, complex or quaternionic.

We choose the normal geodesic such that $c(t)$, $0 \leq t \leq L$ is a minimizing geodesic between the two singular orbits. Thus $c(2n \cdot L) \in G/K_-$ and $c((2n+1) \cdot L) \in G/K_+$. We choose a fixed Q orthonormal basis X_i of \mathfrak{h}^\perp , adapted to the above decomposition. The metric P_t is then described by the functions $f_{ij}(t) = g_t(X_i^*, X_j^*)_{c(t)}$ defined for $t \in \mathbb{R}$. These functions must satisfy certain smoothness conditions at $c(n \cdot L)$, $n \in \mathbb{Z}$ in order for the metric to extend smoothly across the singular orbits. Some of these smoothness conditions are implied by the action of the Weyl group. Recall that the Weyl group W is by definition the stabilizer of the geodesic c modulo its kernel, which by construction is equal to H . If M is compact, W is a dihedral subgroup of $N(H)/H$ and is generated by involutions $w_\pm \in W$ with $w_-(\dot{c}(0)) = -\dot{c}(0)$ and $w_+(\dot{c}(L)) = -\dot{c}(L)$. Thus w_+w_- is a translation by $2L$, and has order $|W|/2$ when W is finite. Since K_- acts linearly on the slice D_- , and the identity component $(K_-)_0$ acts transitively on the unit sphere in D_- , the involution w_- can be represented uniquely as the element $a \in (K_-)_0 \bmod H$ with $a\dot{c}(0) = -\dot{c}(0)$, where $\dot{c}(0) \in D_-$, and similarly for w_+ . Note that W is finite if and only if c is a closed geodesic, and in that case the order $|W|$ is the number of minimal geodesic segments intersecting the regular part. Note also that any non-principal isotropy group along c is of the form $wK_\pm w^{-1}$ for some $w \in N(H)$ representing an element of W .

At a singular point $t_0 = nL$, with stabilizer group K , we define a Q -orthogonal decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p} \quad \text{and thus} \quad \mathfrak{h}^\perp = \mathfrak{p} \oplus \mathfrak{m}.$$

Here \mathfrak{m} can be viewed as the tangent space to the singular orbit G/K at $c(t_0)$. The slice V , i.e. the vector space normal to G/K at $c(t_0)$, can be identified with $\dot{c}(t_0) \oplus \mathfrak{p}$. For this, we send $X \in \mathfrak{p}$ to $\bar{X} = \lim_{t \rightarrow t_0} X^*(c(t))/t$. Note that since K preserves the slice V and acts linearly on it, we have $X^*(c(t)) = t\bar{X} \in V$.

At $t = t_0$ the slice is orthogonal to the orbit, but not in general at nearby points. K acts via the isotropy action $\text{Ad}(K)|_{\mathfrak{m}}$ of G/K on \mathfrak{m} and via the slice representation on V . Thus w_- also acts on V and the tangent space \mathfrak{m} , well defined up to the action of H , and relates the functions. If, e.g., w_- preserves one of the modules \mathfrak{n}_i , then the function f defined by $P_t|_{\mathfrak{n}_i} = f \text{Id}$ must be even (assuming for simplicity that $t_0 = 0$) since $w_-(c(t)) = c(-t)$ and $(\text{Ad}(w)(X))^*(c(t)) = X^*(c(-t))$. Hence $f(t) = f(-t)$.

2. OBSTRUCTIONS

We now discuss the cohomogeneity one manifolds E_{pq} .

Recall that $G = \text{Sp}(1)\text{Sp}(1)$ and

$$K_- = \text{S}_{p,q}^1 = (e^{ipt}, e^{iqt}), \quad K_+ = \{(q, q) \mid q \in \text{Sp}(1)\} \cdot H, \quad \text{and} \quad H = \{(1, \pm 1)\}.$$

with p even and $\text{gcd}(p, q) = 1$, and hence q odd. If $|p \pm q| = 1$ the manifold is an Eschenburg space, see [E], which admits a cohomogeneity one metric whose group diagram is as above and an invariant metric of positive curvature. We will thus assume that $|p \pm q| \neq 1$, and hence $|p \pm q| \geq 3$. In particular, we also have $p \neq 0$, $q \neq 0$.

As in the previous Section, we assume that the normal geodesic c connecting the two singular orbits is parameterized on the interval $[0, L]$. We first determine the Weyl group W . For $t = 0$, the group K_- acts via rotation on the slice D_- with stabilizer group H at $\dot{c}(0)$, i.e. $(e^{ip\pi}, e^{iq\pi}) \in H$. Since p is even and q odd, $w_- = (e^{ip\pi/2}, e^{iq\pi/2})$ is one of $(\pm 1, \pm i) \bmod H$. For $t = L$, $K_+^0 = \text{Sp}(1)$ acts via left multiplication on $D_+ \simeq \mathbb{H}$ and thus $w_+ = (-1, -1)$. Since w_+ lies in the center of G , it acts trivially on $\dot{c}(L)^\perp \subset T_{c(L)}M$ and hence all functions are even at $t = L$. Since $w_+(c(0)) = c(2L)$ we also have $G_{c(2L)} = w_+(K_-)w_+^{-1} = K_-$ and in fact $G_{c(2nL)} = K_-$. On the other hand, $G_{c(3L)} = w_-(K_+)w_-^{-1}$ is different from $G_{c(L)} = K_+$. Notice also that $W = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and hence $c(4L) = c(0)$.

As a basis of $\mathfrak{h}^\perp = \mathfrak{g}$ we choose $X_1 = (i, 0)$, $X_2 = (j, 0)$, $X_3 = (k, 0)$ and $Y_1 = (i, 0)$, $Y_2 = (j, 0)$, $Y_3 = (k, 0)$. Since H lies in the center of G , there are no restrictions for the metric P_t on \mathfrak{h}^\perp . We define

$$\begin{aligned} f_i &= g(X_i^*, X_i^*), & g_i &= g(Y_i^*, Y_i^*), & h_i &= g(X_i^*, Y_i^*) \\ f_{ij} &= g(X_i^*, X_j^*), & g_{ij} &= g(Y_i^*, Y_j^*), & h_{ij} &= g(X_i^*, Y_j^*) \quad \text{for } i \neq j \end{aligned}$$

Notice that at $t = 0$ the only vanishing Killing field is $pX_1^* + qY_1^*$, whereas at $t = L$, only $X_i^* + Y_i^*$, $i = 1, 2, 3$ vanish. For the proof, we will only need the following smoothness conditions at $t = 0$.

LEMMA 2.1. *If the metric is smooth at $t = 0$, then the following hold:*

- (a) $X_1^*, Y_1^*, X_3^*, Y_3^*$ are orthogonal to X_2^* and Y_2^* ,
- (b) f_2 and g_2 are even, and $h_2 = t^k \phi_1(t^2)$, where $k = \min\{|p - q|, |p + q|\}$,
- (c) $g(X_2^*, pX_1^* + qY_1^*) = t^{|p|+2} \phi_2(t^2)$, $g(Y_2^*, pX_1^* + qY_1^*) = t^{|q|+2} \phi_3(t^2)$.

where ϕ_i are smooth functions.

Proof. The proof is similar to the proof of Theorem 6.1 in [GVZ], see also [VZ2]. The decomposition of $\mathfrak{h}^\perp = \mathfrak{g}$ at $t = 0$ is given as follows. \mathfrak{p} is spanned by $pX_1 + qY_1$, and \mathfrak{m}_0 is spanned by $qX_1 - pY_1$ on which K_- acts trivially. Furthermore on $\mathfrak{m}_1 = \text{span}\{X_2, X_3\}$, the stabilizer group K_- acts by $R(2p\theta)$, and on $\mathfrak{m}_2 = \text{span}\{Y_2, Y_3\}$ by $R(2q\theta)$, where $R(\theta)$ is the standard rotation on \mathbb{R}^2 . On the slice K_- acts as $R(2\theta)$ since $H \subset K_-$ is the ineffective kernel of the action, and H has order 2.

Part (a) follows from the fact that the metric is invariant under the action of K_- at $t = 0$. Indeed, since $p \neq \pm q$, all irreducible sub representation of K_- on $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ are inequivalent. Furthermore, \mathfrak{p} is orthogonal to \mathfrak{m} at $t = 0$. Thus the claim follows from Schur's Lemma.

Lemma 6.4 in [GVZ] implies that $f_2 - f_3 = t^{2p} \phi_1(t^2)$ and $f_2 + f_3 = \phi_2(t^2)$ for some smooth functions ϕ_i and hence f_2 is even. Similarly for g_2 . For h_2 we use Lemma 6.5 in [GVZ]. It implies that $h_2 + h_3 = t^{|p-q|} \phi_1(t^2)$ and $h_2 - h_3 = t^{|p+q|} \phi_2(t^2)$ if p and q have the same sign and $h_2 + h_3 = t^{|p+q|} \phi_1(t^2)$ and $h_2 - h_3 = t^{|p-q|} \phi_2(t^2)$ if their signs are opposite. This implies part (b).

For part (c) the proof of [GVZ] Lemma 6.5 needs to be modified as follows, see also [VZ2]. Let e_1, e_2 be a basis of the slice D_- , orthonormal in $g|_{D_-}$, where $e_1 = \dot{c}(0)$, $e_2 = Z \in \mathfrak{p}$ and hence $Z = pX_1 + qY_1$. Furthermore, K_- acts by $R(2\theta)$ on D_- . On the 4 dimensional space $D_- \oplus \mathfrak{m}_1$ the stabilizer group K_- acts by $\text{diag}(R(2\theta), R(2p\theta))$. The metric is given by $g(e_i, e_j) = \delta_{ij}$ and $g(e_1, X_2) = g(e_1, X_3) = 0$ since the geodesic is orthogonal to all orbits. Furthermore, set $r(t) = g(e_2, X_2^*)$ and $s(t) = g(e_2, X_3^*)$. By [VZ2] Lemma 3.4 (b), we see that $r(t) = t^{|p+1|} \phi_1(t^2)$ and $r(t) = t^{|p-1|} \phi_2(t^2)$ and hence $r(t) = t^{|p|+1} \phi(t^2)$. Recall that for $Z \in \mathfrak{p}$ we have $Z^*(c(t)) = te_2$. Hence $g(Z^*, X_2^*) = g(te_2, X_2^*) = t r(t) = t^{|p|+2} \phi(t^2)$. Similarly for $g(Z^*, Y_2^*)$. \square

These conditions hold at $t = 0$, and by the above at $t = 2L$ as well. For us, the important consequence is that h_2 is an odd function with $h_2'(0) = 0$, a property not implied by the Weyl group.

LEMMA 2.2. *If $E_{p,q}$ with $|p \pm q| \geq 3$ carries a metric with nonnegative curvature, then $X_2^* - Y_2^*$ is parallel on $[0, 2L]$.*

Proof. We will use the results in [VZ1]. Recall that a vector space V of Jacobi fields is self adjoint along a geodesic c if $\dim V = \dim M - 1$ and $\langle J_1, J_2 \rangle = \langle J_1', J_2' \rangle$ for all $J_1, J_2 \in V$. Proposition 3.2 in [VZ1] states that a Jacobi field X which belongs to a self adjoint family V of Jacobi fields is parallel on an interval $[t_0, t_1]$, if the following conditions are satisfied:

- (a) $g_{c(t)}(X, X) \neq 0$, $g_{c(t)}(X, X)' = 0$ for $t = t_0$ and $t = t_1$,
- (b) If $Y \in V$ and $g(X(t_1), Y(t_1)) = 0$ then $g(X(t_0), Y(t_0)) = 0$,
- (c) If $Y \in V$ and $Y(t) = 0$ for some $t \in (t_0, t_1)$ then $g(X(t_0), Y(t_0)) = 0$,
- (d) If $Y(t_0) = 0$, then $g(X'(t_0), Y'(t_0)) = 0$.

Since the restriction of the Killing vector fields Z^* , $Z \in \mathfrak{g}$ to the geodesic c are Jacobi fields, a natural self adjoint family is given by $V = \{Z^* \mid Z \in \mathfrak{h}^\perp\}$. For simplicity we use the same letter for $Z \in \mathfrak{g}$ and the action field Z^* on M , and denote the covariant derivative $\nabla_{\dot{c}}Z^*(c(t))$ by Z' .

Let $X = aX_2 + bY_2$ be the vector in $\text{span}\{X_2, Y_2\}$ such that $g_{c(0)}(X, X_2 + Y_2) = 0$. We first show that X is parallel on $[0, 2L]$, and verify the conditions above one at a time.

- (a) X does not vanish in $[0, 2L]$ since the only Jacobi fields that vanish are $pX_1 + qY_1$ at $t = 0$ and $t = 2L$, $X_i + Y_i$ at $t = L$, and by assumption $X \neq X_2 + Y_2$. For the derivative we have $g(X, X)' = a^2 f_2' + 2abh_2' + b^2 g_2'$. But since $|p \pm q| \geq 3$, Lemma 2.1 (b) implies that f_2', h_2' and g_2' vanish at $t = 0$, and $t = 2L$ as well.
- (b) Recall that $G_{c(0)} = G_{c(2L)}$ and hence the Jacobi fields orthogonal to X at $t = 2L$ are exactly the ones that are orthogonal to X at $t = 0$.
- (c) The Jacobi fields that vanish in $(0, 2L)$ belong to \mathfrak{k}_+ . By Lemma 2.1 (a), $X_1 + Y_1$ and $X_3 + Y_3$ are orthogonal to X at $t = 0$, while $X_2 + Y_2$ is orthogonal to X at $t = 0$ by assumption.
- (d) Since $Z = pX_1 + qY_1$ is the only element of V vanishing at $t = 0$, we need to prove that $g_{c(0)}(X', Z') = 0$. By Lemma 2.1 (c), we have $g(X, Z)''(0) = 0$ since $p, q \neq 0$. Clearly $Z(0) = 0$, and since Z is a Jacobi field, $Z''(0) = -R_{c(0)}(Z, \dot{c})\dot{c} = 0$ as well. Thus we have at $t = 0$:

$$0 = g(X, Z)'' = g(X'', Z) + 2g(X', Z') + g(X, Z'') = 2g(X', Z')$$

It follows that for some constants a and b , the Killing vector field $aX_2 + bY_2$ is parallel on the interval $[0, 2L]$.

We now use the fact that if J_1 and J_2 are two Jacobi fields in a self adjoint family with J_1 parallel, then $g(J_1, J_2)' = g(J_1, J_2') = g(J_1', J_2) = 0$. In particular, if J_1 is orthogonal to J_2 at one point, they are orthogonal everywhere. Applying this to the parallel Jacobi field $aX_2 + bY_2$ it follows that it is orthogonal to $X_2 + Y_2$ everywhere since it is at $t = 0$ by construction. Near $t = L$ we have $(X_2^* + Y_2^*)(c(L - t)) = (L - t)(X_2 + Y_2)$ and hence X is orthogonal to $X_2 + Y_2 \in \mathfrak{p}$. But the only vector in $\text{span}\{X_2, Y_2\}$ orthogonal to $X_2 + Y_2$ is $X_2 - Y_2$ due to K_+ invariance. This proves our claim that $X_2 - Y_2$ is parallel. \square

We can now finish the proof of the Theorem in the Introduction. By Lemma 2.2 we have that $\langle X_2 - Y_2, X_2 - Y_2 \rangle = f_2 - 2h_2 + g_2$ is constant. On the other hand, by Lemma 2.1 (b), the functions f_2 and g_2 are even near $t = 0$ and h_2 is odd, and thus $h_2 = 0$. Now notice that this property in fact holds for all $t \in [-L, L]$. Indeed, the geodesic c is a minimizing geodesic from the singular orbit at $t = 0$ to the one at $t = L$. Since G acts transitively on the orbits, and K_- transitively on the unit sphere in D_- , it follows that the normal exponential map of the singular orbit G/K_- is a diffeomorphism on a tubular neighborhood of radius L . Thus the slice in the proof of the smoothness conditions can be defined on a ball of radius L . Altogether, this implies that $h_2(t) = 0$ for $t \in [0, L]$. But then $\langle X_2 + Y_2, X_2 + Y_2 \rangle = f_2 + g_2 = \langle X_2 - Y_2, X_2 - Y_2 \rangle$ is constant as well, which is a contradiction since $X_2 + Y_2$ must vanish at $t = L$.

Remark. (a) For the second family of cohomogeneity one manifolds $E_{p,q}^*$ the principal isotropy group H is trivial and hence, at $t = 0$, K_- acts as $R(\theta)$ on D_- . But this implies that $h_2 + h_3 = t^{2|p-q|}\phi_1(t^2)$ and $h_2 - h_3 = t^{2|p+q|}\phi_1(t^2)$, see the proof of Lemma 2.1. Hence h_2 is even and the

last part of the above proof breaks down. Notice though that the methods imply that the metric is restricted when it has nonnegative curvature: the fields $X_2^* - Y_2^*$ and $X_3^* - Y_3^*$ must be parallel.

(b) Another difference between $E_{p,q}$ and $E_{p,q}^*$ is that in the latter case the Weyl group element w_- is one of $(\pm 1, \pm 1)$ and hence acts as Id on $\mathfrak{p} \oplus \mathfrak{m}$. Furthermore, $W \simeq \mathbb{Z}_2$, and hence c has length $2L$.

(c) We finally observe that at least one of the spaces $E_{p,q}^*$ carries a metric with nonnegative curvature. Indeed, we have an action by $\text{Sp}(1)\text{Sp}(1)$ on $\mathbb{S}^3 \times \mathbb{C}\mathbb{P}^2$ where $(q_1, q_2) \in \text{Sp}(1)\text{Sp}(1)$ acts via $(r, p) \rightarrow (q_1 r q_2^{-1}, q_2 p)$. Here $r \in \mathbb{S}^3 \simeq \text{Sp}(1)$ and $p \in \mathbb{C}\mathbb{P}^2$ with q_2 acting linearly via $\text{Sp}(1) = \text{SU}(2) \subset \text{SU}(3)$. One now easily checks that the group diagram of this action is that of $E_{1,1}^*$, and hence they are equivariantly diffeomorphic.

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